

A sharp Adams inequality in dimension four and its extremal functions

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Abstract

Let Ω be a smooth oriented bounded domain in \mathbb{R}^4 , $H_0^2(\Omega)$ be the Sobolev space, and $\lambda_1(\Omega) = \inf\{\|\Delta u\|_2^2 : u \in H_0^2(\Omega), \|u\|_2 = 1\}$ be the first eigenvalue of the bi-Laplacian operator Δ^2 on Ω . For $\alpha \in [0, \lambda_1(\Omega))$, we define $\|u\|_{2,\alpha}^2 = \|\Delta u\|_2^2 - \alpha\|u\|_2^2$, for $u \in H_0^2(\Omega)$. In this paper, we will prove the following inequality

$$\sup_{u \in H_0^2(\Omega), \|u\|_{2,\alpha} \leq 1} \int_{\Omega} e^{32\pi^2 u(x)^2} dx < \infty.$$

This strengthens a recent result of Lu and Yang [30]. We also show that there exists a function $u^* \in H_0^2(\Omega) \cap C^4(\overline{\Omega})$ such that $\|u^*\|_{2,\alpha} = 1$ and the supremum above is attained by u^* . Our proofs are based on the blow-up analysis method.

1 Introduction

Let Ω be a smooth bounded domain in \mathbb{R}^n . The Sobolev inequality says that the embedding $W_0^{k,p}(\Omega) \hookrightarrow L^{\frac{np}{n-kp}}(\Omega)$ holds if $p < n/k$, where $W_0^{k,p}(\Omega)$ denotes the Sobolev space of functions vanishing on boundary $\partial\Omega$ together their derivatives of order less than $k - 1$. Such inequality plays an important role in many branch of mathematics such as analysis, geometric, partial differential equations, calculus of variations, etc. However, when $p = n/k$ the embedding $W_0^{k,n/k}(\Omega) \hookrightarrow L^\infty(\Omega)$ does not holds. In this case, the Moser–Trudinger and

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Adams inequalities are perfect replacement. The Moser–Trudinger inequality was established independently by Yudovič [50], Pohožaev [35] and Trudinger [41]. This inequality was sharpened by Moser [32] by finding its sharp constant. This sharp form asserts that the existence of a constant $C_0 > 0$ such that

$$\frac{1}{|\Omega|} \int_{\Omega} \exp(\beta |f(x)|^{\frac{n}{n-1}}) dx \leq C_0, \quad (1.1)$$

for any $\beta \leq \beta_0 = n\omega_{n-1}^{1/(n-1)}$ where ω_{n-1} denotes the surface area of the unit sphere of \mathbb{R}^n , for any bounded domain Ω and for any function $f \in W_0^{1,n}(\Omega)$ with $\|\nabla f\|_n \leq 1$. If $\beta > \beta_0$ then the above inequality does not hold with uniform C_0 independent of f . Moser–Trudinger is a crucial tool in studying the partial differential equation inequality with exponential nonlinearity. Because of its importance, there are many generalization of Moser–Trudinger inequality, such as Moser–Trudinger inequality on Heisenberg group, on complex sphere or on compact Riemannian manifold [5, 6, 22]. It was also extended to entire Euclidean space by Ruf [37] for dimension two and by Li and Ruf [25] for any dimension or entire Heisenberg group by Lam and Lu [18], or on hyperbolic space by Wang and Ye [42]. In [39], Tian and Zhu proved a Moser–Trudinger type inequality for almost plurisubharmonic functions on any Kähler–Einstein manifolds with positive curvature.

The existence of the extremal function for Moser–Trudinger inequality was first proved by Carleson and Chang [4] for the unit ball in \mathbb{R}^n . In [13], Flucher proved the existence of extremal function for Moser–Trudinger inequality for any smooth domain in \mathbb{R}^2 . This result was then extended to any dimension by Lin [28]. The existence of extremal function for Moser–Trudinger inequality on compact Riemannian manifold was studied by Li [23]. We refer the reader to [7, 8, 10, 11, 24, 25, 37, 42, 44, 45, 47–49] for more existence results of extremal functions for Moser–Trudinger type inequalities.

Suggesting by the concentration–compactness principle due to Lions [29], Adimurthi and Druet established in [2] the following generalization of Moser–Trudinger inequality on any bounded domain $\Omega \subset \mathbb{R}^2$

$$\sup_{u \in H_0^1(\Omega), \|\nabla u\|_2 \leq 1} \int_{\Omega} e^{4\pi(1+\alpha\|u\|_2^2)u^2} dx < \infty, \quad (1.2)$$

for any $0 \leq \alpha < \lambda(\Omega)$, where $\lambda(\Omega) = \inf_{u \in H_0^1(\Omega), \|u\|_2 \leq 1} \|\nabla u\|_2^2$ is the first eigenvalue of Laplace operator $-\Delta$. The existence of extremal function for (1.2) was proved by Yang in [44]. This result was extended by Yang [45, 46] to the cases of high dimension and compact Riemannian surfaces, by Lu and Yang [31] and Zhu [51] to the version of L^- -norm, by Souza and do Ó [10, 11] to the whole Euclidean space, and by Tintarev [40] to the following form

$$\sup_{u \in H_0^1(\Omega), \|\nabla u\|_2^2 - \alpha\|u\|_2 \leq 1} \int_{\Omega} e^{4\pi u^2} dx < \infty, \quad (1.3)$$

with $0 \leq \alpha < \lambda(\Omega)$. Evidently, (1.3) implies (1.2). In [47], Yang generalized (1.3) to the cases that large eigenvalues are involved, as well as to the manifold case. The existence

of extremal functions for (1.3) also obtained in [47]. In [48], Yang and Zhu studied the singular version of (1.3). They proved the existence of extremal functions for the following singular Moser–Trudinger inequality

$$\sup_{u \in H_0^1(\Omega), \|\nabla u\|_2^2 - \alpha \|u\|_2 \leq 1} \int_{\Omega} \frac{e^{4\pi(1-\beta)u^2}}{|x|^{2\beta}} dx < \infty, \quad \alpha < \lambda(\Omega) \quad (1.4)$$

where Ω is a smooth bounded domain in \mathbb{R}^2 containing the origin in its interior and $0 \leq \beta < 1$. The same existence result for the singular Moser–Trudinger inequality on whole Euclidean space was recently proved by Yang and Zhu in [49].

Adams inequality is the version of higher order of derivatives of Moser–Trudinger inequality. The study of this inequality was started by the work of Adams [1]. To state Adams inequality, we use the symbol $\nabla^m u$ with m is a positive integer, to denote the m^{th} order gradient for $u \in C^m$, the class of m^{th} order differentiable functions,

$$\nabla^m u = \begin{cases} \Delta^{m/2} u & \text{if } m \text{ even,} \\ \nabla \Delta^{(m-1)/2} u & \text{if } m \text{ odd,} \end{cases}$$

where ∇ and Δ denotes the usual gradient operator and usual Laplacian respectively. Adams proved in [1] that for any positive integer m less than n , there exists a constant $C_0(n, m)$ such that for any bounded domain $\Omega \subset \mathbb{R}^n$, it holds

$$\sup_{u \in W_0^{m, \frac{n}{m}}(\Omega), \|\nabla^m u\|_{\frac{n}{m}} \leq 1} \frac{1}{|\Omega|} \int_{\Omega} \exp(\beta |u|^{\frac{n}{n-m}}) dx \leq C_0(n, m), \quad (1.5)$$

for any $\beta \leq \beta(n, m)$ with

$$\beta(n, m) = \begin{cases} \frac{n}{\omega_{n-1}} \left[\frac{\pi^{n/2} 2^m \Gamma(\frac{m+1}{2})}{\Gamma(\frac{n-m+1}{2})} \right]^{\frac{n}{n-m}} & \text{if } m \text{ odd,} \\ \frac{n}{\omega_{n-1}} \left[\frac{\pi^{n/2} 2^m \Gamma(\frac{m}{2})}{\Gamma(\frac{n-m}{2})} \right]^{\frac{n}{n-m}} & \text{if } m \text{ even.} \end{cases}$$

Furthermore, for $\beta > \beta(n, m)$ the supremum above will be infinite. Notice that when $m = 1$, (1.5) reduces to Moser–Trudinger inequality (1.1).

Remark that the work of Moser and of Carleson and Chang was based on the rearrangement argument to reduce problem to the one-dimensional problem. However, we can not adapt this symmetrization technique in the case $m \geq 2$ since we do not know whether the $L^{\frac{n}{m}}$ norm of the m^{th} gradient of a function decreases under the rearrangement operator. In order to establish (1.5), Adams use the representation of u in terms of its gradient function $\nabla^m u$ using a convolution operator, and then apply O’Neil’s idea [34] of rearrangement of convolution of two functions together with the idea which originally goes back to Garcia. Such an argument avoids in dealing with the issue of $L^{\frac{n}{m}}$ norm preserving of the gradient of the rearranged functions. This idea has also been developed to derive the sharp Adams inequality on Riemannian manifolds without boundary by Fontana [14], on the measure

spaces by Fontana and Morpurgo [15]. The sharp Adams inequality was also generalized to whole Euclidean space in the works of Fontana and Morpurgo [16], of Lam and Lu [19, 20] and of Ruf and Sani [38]. The sharp Adams inequality was recently established on the hyperbolic spaces by Karmakar and Sandeep [17].

It remains an open problem whether Adams inequality has an extremal function. Unlike in Moser–Trudinger inequality with first order derivative, we can not adapt Carleson–Chang’s idea [4] of symmetrization to establish the existence of extremal function for inequalities of higher order derivatives. It is still a rather difficult problem to answer the above question in the most generality. One interesting case of the above question when $n = 4$ and $m = 2$ was addressed in [30]. Let $\Omega \subset \mathbb{R}^4$ denote a smooth oriented bounded domain, $H_0^2(\Omega)$ denote the Sobolev space which is completion of the space of compactly supported smooth functions in Ω under the Dirichlet norm $\|u\|_{H_0^2(\Omega)} = \|\Delta u\|_2$. Then Adams inequality in the case $n = 4$ and $m = 2$ states that

$$\sup_{u \in H_0^2(\Omega), \|\Delta u\|_2 \leq 1} \int_{\Omega} e^{\gamma u^2} dx < \infty, \quad (1.6)$$

for any $\gamma \leq 32\pi^2$. The existence of extremal function for inequality (1.6) was proved by Lu and Yang [30]. Even, Lu and Yang established in [30] an improvement of (1.6) in spirit of Adimurthi and Druet (for improvement of Moser–Trudinger inequality (1.2)). Let $\lambda_1(\Omega)$ denote the first eigenvalue of the bi-Laplacian operator Δ^2 on Ω , i.e.,

$$\lambda_1(\Omega) = \inf_{u \in H_0^2(\Omega), u \neq 0} \frac{\|\Delta u\|_2^2}{\|u\|_2^2}.$$

An easy application of the variational method shows that $\lambda_1(\Omega) > 0$ and is attained. It was proved by Lu and Yang that

$$\sup_{u \in H_0^2(\Omega), \|\Delta u\|_2 \leq 1} \int_{\Omega} e^{32\pi^2 q(\|u\|_2^2) u^2} dx < \infty, \quad (1.7)$$

where $q(t) = 1 + a_1 t + \dots + a_k t^k$, $k \geq 1$ is a polynomial of order k in \mathbb{R} with $0 \leq a_1 < \lambda_1(\Omega)$, $0 \leq a_2 \leq \lambda_1(\Omega)a_1$, ..., $0 \leq a_k \leq \lambda_1(\Omega)a_{k-1}$. Furthermore, if $a_1 \geq \lambda_1(\Omega)$ then the supremum above will be infinite.

The existence of extremal functions for inequality (1.7) was also studied in [30]. It was proved that there exists a strictly positive constant $\epsilon_0 < \lambda_1(\Omega)$ depending only on Ω such that when $0 \leq a_1 < \lambda_1(\Omega)$, $0 \leq a_2 \leq \lambda_1(\Omega)a_1$, ..., $0 \leq a_k \leq \lambda_1(\Omega)a_{k-1}$, we can find $u^* \in H_0^2(\Omega) \cap C^4(\overline{\Omega})$ such that $\|\Delta u^*\|_2 = 1$ and

$$\int_{\Omega} e^{32\pi^2 q(\|u^*\|_2^2) u^{*2}} dx = \sup_{u \in H_0^2(\Omega), \|\Delta u\|_2 \leq 1} \int_{\Omega} e^{32\pi^2 q(\|u\|_2^2) u^2} dx.$$

Obviously, this implies the existence of extremal functions for Adams inequality (1.6).

The first aim of this paper is to strengthen Adams inequality (1.6) in the spirit of Tintarev for the improvement of Moser–Trudinger inequality (1.3). To do this, let us define for any $0 \leq \alpha < \lambda_1(\Omega)$,

$$\|u\|_{2,\alpha}^2 = \|\Delta u\|_2^2 - \alpha \|u\|_2^2, \quad u \in H_0^2(\Omega).$$

Note that $\|\cdot\|_{2,\alpha}$ is a norm on $H_0^2(\Omega)$ which is equivalent to $\|\cdot\|_{H_0^2(\Omega)}$. In this paper, we will prove the following inequality.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^4$ be a smooth, oriented bounded domain, $\lambda_1(\Omega)$ be the first eigenvalue of bi-Laplacian operator Δ^2 on Ω . Then for any α with $0 \leq \alpha < \lambda_1(\Omega)$, we have*

$$\sup_{u \in H_0^2(\Omega), \|u\|_{2,\alpha} \leq 1} \int_{\Omega} e^{32\pi^2 u^2} dx < \infty. \quad (1.8)$$

Remark that when $\alpha = 0$, (1.8) reduces to (1.6). Moreover, for any $0 \leq \alpha < \lambda_1(\Omega)$ and $u \in H_0^2(\Omega)$ such that $\|\Delta u\|_2 \leq 1$ denote $v = u/\|u\|_{2,\alpha}$, then $u^2 \leq v^2$, and $\|v\|_{2,\alpha} = 1$, thus (1.8) is indeed stronger than Adams inequality (1.6). The next result shows that (1.8) is stronger than the inequality of Lu and Yang (1.7).

Proposition 1.2. *Theorem 1.1 implies the inequality (1.7).*

The second result of this paper is the existence of the extremal functions for the inequality (1.8). More precisely, we prove the following result.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^4$ be a smooth, oriented bounded domain, $\lambda_1(\Omega)$ be the first eigenvalue of bi-Laplacian operator Δ^2 on Ω . Then for any α with $0 \leq \alpha < \lambda_1(\Omega)$, there exists $u^* \in H_0^2(\Omega) \cap C^4(\overline{\Omega})$ such that $\|u^*\|_{2,\alpha} = 1$ and*

$$\int_{\Omega} e^{32\pi^2 u^{*2}} dx = \sup_{u \in H_0^2(\Omega), \|u\|_{2,\alpha} \leq 1} \int_{\Omega} e^{32\pi^2 u^2} dx.$$

Note that when $\alpha = 0$ we obtain the existence of extremal function for Adams inequality (1.6) which was already proved in [30]. Although, our inequality (1.8) is stronger than the one of Lu and Yang (1.7), however the existence result in Theorem 1.3 does not imply the existence result for the inequality (1.7). Also, contrary with the existence result of Lu and Yang, our Theorem 1.3 gives the existence of extremal function for the inequality (1.8) for any $0 \leq \alpha < \lambda_1(\Omega)$.

We conclude this introduction by mentioning about the method of proof of our main Theorems. As usually, our method is based on the blow-up analysis method. We first establish a concentration-compactness lemma of Lion's type and using it to prove the existence of $u_{\epsilon} \in H_0^2(\Omega) \cap C^4(\overline{\Omega})$, $\epsilon \in (0, 32\pi^2)$ such that $\|u_{\epsilon}\|_{2,\alpha} = 1$ and

$$\int_{\Omega} e^{32\pi^2 u_{\epsilon}^2} dx = \sup_{u \in H_0^2(\Omega), \|u\|_{2,\alpha} \leq 1} \int_{\Omega} e^{(32\pi^2 - \epsilon)u^2} dx.$$

Thus, the Euler–Lagrange equation of u_ϵ is given by

$$\begin{cases} \Delta^2 u_\epsilon = \frac{1}{\lambda_\epsilon} e^{(32\pi^2 - \epsilon)u_\epsilon^2} u_\epsilon + \alpha u_\epsilon & \text{in } \Omega, \\ \|u_\epsilon\|_{2,\alpha} = 1, u_\epsilon = \frac{\partial u_\epsilon}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ \lambda_\epsilon = \int_\Omega e^{(32\pi^2 - \epsilon)u_\epsilon^2} u_\epsilon^2 dx, \end{cases}$$

where ν denotes the outward unit normal vector to $\partial\Omega$. Without loss of generality, let $c_\epsilon = \max_{\overline{\Omega}} |u_\epsilon| = u_\epsilon(x_\epsilon)$. If c_ϵ is bounded, by the standard regularity theory we obtain $u_\epsilon \rightarrow u^*$ in $C^4(\overline{\Omega})$ hence finishes our proof. If $c_\epsilon \rightarrow \infty$ (namely, the blow-up occurs) and $x_\epsilon \rightarrow p \in \overline{\Omega}$, by using Pohozaev type identity and elliptic estimates, we exclude the case $p \in \partial\Omega$. We also show that $c_\epsilon u_\epsilon$ converges to some Green function weakly in $H_0^2(\Omega)$ which then immediately leads to Theorem 1.1. We also prove an upper bound for functional $\int_\Omega e^{32\pi^2 u^2} dx$ when blow-up occurs by using some capacity estimates. By constructing a sequence of test functions, we exclude the blow-up phenomena for the maximizing sequence of functional $\int_\Omega e^{32\pi^2 u^2} dx$. This leads to the existence result in Theorem 1.3. We emphasize here that in our proof below, we do not require the sharp Adams inequality (i.e., $\gamma = 32\pi^2$ in (1.6)), but only require the subcritical Adams inequality (i.e., $\gamma < 32\pi^2$ in (1.6)). We also would like to mention here that blow-up analysis technique have been already employed by numerous authors in relevant but quite different setting in dealing with Sobolev inequalities instead of Moser–Trudinger inequality. We refer the interested reader to the works [3, 10–12, 21–25, 30, 42, 44–49], etc.

The rest of this paper is organized as follows. In section §2, we give the existence of maximizers for subcritical functional. In section §3 we analyse the asymptotic behavior of those maximizers functions. In section §4, we obtain an upper bound for the critical functional under the assumption that blow-up occurs in the interior of Ω by using some capacity estimates. We exclude the boundary bubble in section §5. The proof of Theorem 1.1 and Proposition 1.2 is given in section §6. In section §7, we construct a sequence of test functions to conclude the existence of extremal function for the critical functional and thus give the proof of Theorem 1.3.

2 Extremals for the subcritical Adams inequality

For any $\epsilon \in (0, 32\pi^2)$, let us consider the subcritical problems

$$C_\epsilon = \sup_{u \in H_0^2(\Omega), \|u\|_{2,\alpha} \leq 1} \int_\Omega e^{(32\pi^2 - \epsilon)u^2} dx. \quad (2.1)$$

In this section, we mainly prove that $C_\epsilon < \infty$ and the subcritical problem (2.1) is attained. Noting that the existence of such extremals is nontrivial. In the proof, we need the following Lion’s type [29] concentration–compactness principle.

Proposition 2.1. *Let $\{u_j\}_j \subset H_0^2(\Omega)$ be a sequence of functions such that $\|u_j\|_{2,\alpha} = 1$ and $u_j \rightharpoonup u_0$ weakly in $H_0^2(\Omega)$. Then for any $p < (1 - \|u_0\|_{2,\alpha}^2)^{-1}$,*

$$\limsup_{j \rightarrow \infty} \int_{\Omega} e^{32\pi^2 p u_j^2} dx < \infty.$$

Proof. By Rellich–Kondrachov theorem, we have $\|u_j\|_2 \rightarrow \|u_0\|_2$ as $j \rightarrow \infty$. Denote

$$v_j = \frac{u_j}{\|\Delta u_j\|_2} = \frac{u_j}{(1 + \alpha\|u_j\|_2^2)^{1/2}},$$

then $\|\Delta v_j\|_2 = 1$ and $v_j \rightharpoonup v_0 = u_0/(1 + \alpha\|u_0\|_2^2)^{1/2}$ weakly in $H_0^2(\Omega)$. Applying the Lions type concentration–compactness principle of Lu and Yang (see Proposition 3.1 in [30]), we have

$$\limsup_{j \rightarrow \infty} \int_{\Omega} e^{32\pi^2 q v_j^2} dx < \infty \quad (2.2)$$

for any $q < 1/(1 - \|\Delta v_0\|_2^2)$. For any $p < 1/(1 - \|u_0\|_{2,\alpha}^2)$ we have

$$\lim_{j \rightarrow \infty} p\|\Delta u_j\|_2^2 = p(1 + \alpha\|u_0\|_2^2) < \frac{1 + \alpha\|u_0\|_2^2}{1 + \alpha\|u_0\|_2^2 - \|\Delta u_0\|_2^2} = \frac{1}{1 - \|\Delta v_0\|_2^2}.$$

This implies the existence of j_0 and $q < 1/(1 - \|\Delta v_0\|_2^2)$ such that

$$p\|\Delta u_j\|_2^2 \leq q < \frac{1}{1 - \|\Delta v_0\|_2^2}, \quad \forall j \geq j_0.$$

Thus by (2.2), we get

$$\limsup_{j \rightarrow \infty} \int_{\Omega} e^{32\pi^2 p u_j^2} dx = \limsup_{j \rightarrow \infty} \int_{\Omega} e^{32\pi^2 p \|\Delta u_j\|_2^2 v_j^2} dx \leq \limsup_{j \rightarrow \infty} \int_{\Omega} e^{32\pi^2 q v_j^2} dx < \infty,$$

as our desire. □

Our existence result is given in the following proposition.

Proposition 2.2. *For any $\epsilon \in (0, 32\pi^2)$, we have $C_\epsilon < \infty$ and there exists $u_\epsilon \in H_0^2(\Omega)$ such that $\|u_\epsilon\|_{2,\alpha} = 1$ and*

$$C_\epsilon = \int_{\Omega} e^{(32\pi^2 - \epsilon)u_\epsilon^2} dx.$$

Note that $32\pi^2 - \epsilon$ can be replaced by any sequence $\{\rho_\epsilon\}_\epsilon$ with $\rho_\epsilon \uparrow 32\pi^2$.

Proof. Let $\{u_j\}_j \subset H_0^2(\Omega)$ be a sequence of functions with $\|u_j\|_{2,\alpha} = 1$ and

$$\lim_{j \rightarrow \infty} \int_{\Omega} e^{(32\pi^2 - \epsilon)u_j^2} dx = C_\epsilon.$$

Since $\alpha \in [0, \lambda_1(\Omega))$ then

$$1 = \|u_j\|_{2,\alpha}^2 \geq \left(1 - \frac{\alpha}{\lambda_1(\Omega)}\right) \|\Delta u_j\|_2^2.$$

Thus $\{u_j\}_j$ is bounded in $H_0^2(\Omega)$. Up to a subsequence, we can assume that $u_j \rightharpoonup u_\epsilon$ weakly in $H_0^2(\Omega)$, $u_j \rightarrow u_\epsilon$ in $L^p(\Omega)$ for any $1 < p < \infty$ and $u_j \rightarrow u_\epsilon$ a.e., in Ω . If $u_\epsilon = 0$, then by Rellich–Kondrachov theorem, we have $\|u_j\|_2 \rightarrow 0$ as $j \rightarrow \infty$. Define

$$v_j = \frac{u_j}{\|\Delta u_j\|_2} = \frac{u_j}{(1 + \alpha\|u_j\|_2^2)^2},$$

then $\|\Delta v_j\|_2 = 1$ and $v_j \rightarrow 0$ a.e., in Ω . Since

$$\lim_{j \rightarrow \infty} (32\pi^2 - \epsilon)(1 + \alpha\|u_j\|_2^2) = 32\pi^2 - \epsilon,$$

and by Adams inequality

$$\sup_{j \geq 1} \int_{\Omega} e^{32\pi^2 v_j^2} dx < \infty,$$

then there exists $p > 1$ such that

$$\sup_j \int_{\Omega} e^{(32\pi^2 - \epsilon) p u_j^2} dx < \infty.$$

Thus, since $v_j \rightarrow 0$ a.e., in Ω , by letting $j \rightarrow \infty$ we get

$$\lim_{j \rightarrow \infty} \int_{\Omega} e^{(32\pi^2 - \epsilon) u_j^2} dx = |\Omega|,$$

which is impossible. Hence $u_\epsilon \not\equiv 0$ and

$$0 < \|\Delta u_\epsilon\|_2^2 - \alpha\|u_\epsilon\|_2^2 \leq \liminf_{j \rightarrow \infty} \|u_j\|_{2,\alpha}^2 \leq 1,$$

It follows from Proposition 2.1 that

$$\sup_{j \geq 1} \int_{\Omega} e^{32\pi^2 p u_j^2} dx < \infty$$

for any $p < 1/(1 - \|u_\epsilon\|_{2,\alpha}^2)$. This together $u_j \rightarrow u_\epsilon$ a.e., in Ω implies

$$\lim_{j \rightarrow \infty} \int_{\Omega} e^{(32\pi^2 - \epsilon) u_j^2} dx = \int_{\Omega} e^{(32\pi^2 - \epsilon) u_\epsilon^2} dx.$$

This shows that $C_\epsilon < \infty$. Obviously, we must have $\|u_\epsilon\|_{2,\alpha} = 1$. Hence u_ϵ is a maximizer for C_ϵ . \square

An easy computation shows that the Euler–Lagrange equation of u_ϵ is given by

$$\begin{cases} \Delta^2 u_\epsilon = \frac{1}{\lambda_\epsilon} e^{\alpha_\epsilon u_\epsilon^2} u_\epsilon + \alpha u_\epsilon & \text{in } \Omega \\ \|u_\epsilon\|_{2,\alpha} = 1, \quad u_\epsilon = \frac{\partial u_\epsilon}{\partial \nu} = 0 & \text{on } \partial\Omega \\ \alpha_\epsilon = 32\pi^2 - \epsilon, \quad \lambda_\epsilon = \int_\Omega e^{\alpha_\epsilon u_\epsilon^2} u_\epsilon^2 dx. \end{cases} \quad (2.3)$$

Lemma 2.3. *It holds $\liminf_{\epsilon \rightarrow 0} \lambda_\epsilon > 0$.*

Proof. Using the inequality $e^t \leq 1 + te^t$ for $t \geq 0$, we get

$$C_\epsilon = \int_\Omega e^{\alpha_\epsilon u_\epsilon^2} dx \leq |\Omega| + \alpha_\epsilon \lambda_\epsilon. \quad (2.4)$$

It is evident that

$$\limsup_{\epsilon \rightarrow 0} C_\epsilon \leq \sup_{u \in H_0^2(\Omega), \|u\|_{2,\alpha}=1} \int_\Omega e^{32\pi^2 u^2} dx.$$

For any $u \in H_0^2(\Omega)$ with $\|u\|_{2,\alpha} = 1$, by Fatou's lemma we have

$$\int_\Omega e^{32\pi^2 u^2} dx \leq \liminf_{\epsilon \rightarrow 0} \int_\Omega e^{\alpha_\epsilon u^2} dx \leq \liminf_{\epsilon \rightarrow 0} C_\epsilon.$$

Taking the supremum over all such functions u , we have

$$\sup_{u \in H_0^2(\Omega), \|u\|_{2,\alpha}=1} \int_\Omega e^{32\pi^2 u^2} dx \leq \liminf_{\epsilon \rightarrow 0} C_\epsilon.$$

Thus we have shown that

$$\lim_{\epsilon \rightarrow 0} C_\epsilon = \sup_{u \in H_0^2(\Omega), \|u\|_{2,\alpha}=1} \int_\Omega e^{32\pi^2 u^2} dx > |\Omega|. \quad (2.5)$$

Combining (2.4) and (2.5) together we obtain the desired estimate. \square

3 Asymptotic behavior of extremals for subcritical functionals

The crucial tool in studying the regularity of higher order equations is the Green's representation formula. The Green function $G(x, y)$ for Δ^2 under the Dirichlet condition is the solution of

$$\Delta^2 G(x, y) = \delta_x(y) \quad \text{in } \Omega, \quad G(x, y) = \frac{\partial G(x, y)}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \quad (3.1)$$

All functions $u \in H_0^2(\Omega) \cap C^4(\overline{\Omega})$ satisfying $\Delta^2 u = f$ can be represented by

$$u(x) = \int_\Omega G(x, y) f(y) dy.$$

We will need the following useful estimates [9] for G in the analysis below

$$|G(x, y)| \leq C \ln(2 + |x - y|^{-1}), \quad |\nabla^i G(x, y)| \leq C |x - y|^{-i}, \quad i \geq 1, \quad (3.2)$$

for some constant $C > 0$ and for all $x, y \in \Omega$, $x \neq y$.

Denote $c_\epsilon = \max_{x \in \Omega} |u_\epsilon(x)| = |u_\epsilon(x_\epsilon)|$ for $x_\epsilon \in \Omega$. If c_ϵ is bounded, then applying the standard regularity to (2.3) we obtain $u_\epsilon \rightarrow u^*$ in $C^4(\overline{\Omega})$ for some $u^* \in H_0^2(\Omega) \cap C^4(\overline{\Omega})$ with $\|u^*\|_{2,\alpha} = 1$. This then implies

$$\int_{\Omega} e^{32\pi^2 u^{*2}} dx = \sup_{u \in H_0^2(\Omega), \|u\|_{2,\alpha}=1} \int_{\Omega} e^{32\pi^2 u^2} dx,$$

which leads to our desired results.

In the sequel, we assume that $c_\epsilon \rightarrow \infty$. Without loss of generality we assume that

$$c_\epsilon = u_\epsilon(x_\epsilon) = \max_{x \in \Omega} |u_\epsilon(x)| \rightarrow \infty, \quad x_\epsilon \in \Omega, \quad x_\epsilon \rightarrow p \in \overline{\Omega} \quad \text{as } \epsilon \rightarrow 0. \quad (3.3)$$

As in [30], we call p the blow-up point. Here and in the sequel, we do not distinguish sequence and subsequence, the reader can understand it from the context.

Since $\|u_\epsilon\|_{2,\alpha} = 1$ and $\alpha < \lambda_1(\Omega)$ then

$$\|\Delta u_\epsilon\|_2^2 \leq 1 + \frac{\alpha}{\lambda_1(\Omega)},$$

hence u_ϵ is bounded in $H_0^2(\Omega)$, we can assume that $u_\epsilon \rightharpoonup u_0$ weakly in $H_0^2(\Omega)$, $u_\epsilon \rightarrow u_0$ in $L^s(\Omega)$ for any $1 < s < \infty$ and $u_\epsilon \rightarrow u_0$ a.e., in Ω . If $u_0 \not\equiv 0$, then by Lions type concentration-compactness principle (Proposition 2.1), there is $p > 1$ such that

$$\sup_{\epsilon > 0} \int_{\Omega} e^{32\pi^2 p u_\epsilon^2} dx < \infty.$$

Hence $e^{\alpha_\epsilon u_\epsilon}$ is bounded in $L^r(\Omega)$ for some $r > 1$ provided that ϵ is small enough. Applying the standard regularity theory to (2.3), we obtain the boundedness of c_ϵ which is contradiction with (3.3). Hence, we have

$$\begin{cases} u_\epsilon \rightharpoonup 0 & \text{weakly in } H_0^2(\Omega), \\ u_\epsilon \rightarrow 0 & \text{in } L^r(\Omega) \text{ for any } r > 1, \text{ and a.e., in } \Omega, \\ \alpha_\epsilon \rightarrow 32\pi^2. \end{cases} \quad (3.4)$$

In the rest of this section we focus on the case $p \in \Omega$ (the case $p \in \partial\Omega$ will be treated below in §5). We claim that

$$|\Delta u_\epsilon|^2 dx \rightharpoonup \delta_p \quad \text{in the sense of measure.} \quad (3.5)$$

Indeed, if (3.5) does not hold. Since $\|\Delta u_\epsilon\|_2^2 = 1 + \alpha\|u_\epsilon\|_2^2 \rightarrow 1$ as $\epsilon \rightarrow 0$, we can find $r > 0$ and $\eta > 0$ such that $B_r(p) \subset \Omega$ and

$$\limsup_{\epsilon \rightarrow 0} \int_{B_r(p)} |\Delta u_\epsilon|^2 dx \leq 1 - \eta.$$

From Sobolev embedding theorem and (3.4), we have $\nabla u_\epsilon \rightarrow 0$ strongly in $L^2(\Omega)$. Let $\phi \in C_0^\infty(B_r(p))$ be a cut-off function with $0 \leq \phi \leq 1$ and $\phi = 1$ on $B_{r/2}(p)$. We have

$$\limsup_{\epsilon \rightarrow 0} \int_{B_r(p)} |\Delta(\phi u_\epsilon)|^2 dx \leq 1 - \eta.$$

By Adams inequality, $e^{\alpha_\epsilon \phi^2 u_\epsilon^2}$ is bounded in $L^{2/(2-\eta)}(\Omega)$ and hence $e^{\alpha_\epsilon u_\epsilon^2}$ is bounded in $L^{2/(2-\eta)}(B_{r/2}(p))$ provided that ϵ is small enough. Applying the standard regularity theory to (2.3), we have that u_ϵ is bounded in $C^1(\overline{B_{r/4}(p)})$. This contradicts our assumption (3.3). Hence, we obtain (3.5). In fact, we have shown that there is no other blow-up point if p lies in Ω and $\|u_\epsilon\|_{2,\alpha} = 1$.

To proceed, we introduce the following quantities

$$b_\epsilon = \frac{\lambda_\epsilon}{\int_\Omega |u_\epsilon| e^{\alpha_\epsilon u_\epsilon^2} dx}, \quad \tau = \lim_{\epsilon \rightarrow 0} \frac{c_\epsilon}{b_\epsilon}, \quad \sigma = \lim_{\epsilon \rightarrow 0} \frac{\int_\Omega u_\epsilon e^{\alpha_\epsilon u_\epsilon^2} dx}{\int_\Omega |u_\epsilon| e^{\alpha_\epsilon u_\epsilon^2} dx}. \quad (3.6)$$

Note that $\tau \geq 1$ or $\tau = \infty$, $|\sigma| \leq 1$. We will show that $\sigma = 1$ at the end of this section.

Let

$$r_\epsilon^4 = \frac{\lambda_\epsilon}{c_\epsilon^2} e^{-\alpha_\epsilon c_\epsilon^2}, \quad \Omega_\epsilon = \{x \in \mathbb{R}^4 : x_\epsilon + r_\epsilon x \in \Omega\}.$$

We will show that r_ϵ converges to zero rapidly. Indeed, for any $0 < \gamma < 32\pi^2$, we have

$$r_\epsilon^4 c_\epsilon^2 e^{\gamma c_\epsilon^2} = e^{(\gamma - \alpha_\epsilon) c_\epsilon^2} \int_\Omega u_\epsilon^2 e^{\alpha_\epsilon u_\epsilon^2} dx \leq \int_\Omega u_\epsilon^2 e^{\gamma u_\epsilon^2} dx \rightarrow 0, \quad (3.7)$$

here we used Hölder inequality, (3.4) and the fact $0 < \gamma < 32\pi^2$. In particular, $r_\epsilon \rightarrow 0$ and $\Omega_\epsilon \rightarrow \mathbb{R}^4$ as $\epsilon \rightarrow 0$. We next define two sequences of functions on Ω_ϵ by

$$\psi_\epsilon(x) = \frac{u_\epsilon(x_\epsilon + r_\epsilon x)}{c_\epsilon}, \quad \varphi_\epsilon(x) = b_\epsilon(u_\epsilon(x_\epsilon + r_\epsilon x) - c_\epsilon) = b_\epsilon c_\epsilon(\psi_\epsilon(x) - 1).$$

Our next goal is to understand the asymptotic behavior of ψ_ϵ and φ_ϵ . Evidently, $|\psi_\epsilon| \leq 1$ and

$$\Delta^2 \psi_\epsilon(x) = r_\epsilon^4 \left(\frac{1}{\lambda_\epsilon} \psi_\epsilon(x) e^{\alpha_\epsilon u_\epsilon(x_\epsilon + r_\epsilon x)^2} + \alpha \psi_\epsilon(x) \right).$$

Thus, for any $R > 0$ and $x \in B_R(0)$ we have

$$|\Delta^2 u_\epsilon(x)|^2 \leq \frac{1}{c_\epsilon^2} + \alpha r_\epsilon^4 \rightarrow 0,$$

and

$$\int_{B_R(0)} |\Delta \psi_\epsilon|^2 dx = \frac{1}{c_\epsilon^2} \int_{B_{r_\epsilon R}(x_\epsilon)} |\Delta u_\epsilon|^2 dx \rightarrow 0.$$

These estimates and the standard regularity theory give $\psi_\epsilon \rightarrow \psi$ in $C_{\text{loc}}^4(\mathbb{R}^4)$ with $\Delta \psi = 0$ in \mathbb{R}^4 . Note that $|\psi_\epsilon| \leq 1$ and $\psi_\epsilon(0) = 1$, then $|\psi| \leq 1$ and $\psi(0) = 1$. Using Liouville theorem, we conclude that $\psi \equiv 1$ in \mathbb{R}^4 . Thus, we have proved that

Lemma 3.1. *It holds $\psi_\epsilon \rightarrow 1$ in $C_{\text{loc}}^4(\mathbb{R}^4)$.*

We next investigate the convergence of φ_ϵ .

Lemma 3.2. *Let τ be defined in (3.6). Then $\varphi_\epsilon \rightarrow \varphi$ in $C_{\text{loc}}^4(\mathbb{R}^4)$, where*

$$\varphi(x) = \begin{cases} \frac{1}{16\pi^2\tau} \ln \frac{1}{1+\frac{\tau}{\sqrt{6}}|x|^2} & \text{if } \tau < \infty, \\ 0 & \text{if } \tau = \infty, \end{cases} \quad (3.8)$$

for $x \in \mathbb{R}^4$.

Proof. Using Green representation formula, we have

$$u_\epsilon(x) = \int_{\Omega} G(x, y) \left(\frac{1}{\lambda_\epsilon} e^{\alpha_\epsilon u_\epsilon(y)^2} u_\epsilon(y) + \alpha u_\epsilon(y) \right) dy$$

hence

$$\nabla^i u_\epsilon(x) = \int_{\Omega} \nabla_x^i G(x, y) \left(\frac{1}{\lambda_\epsilon} e^{\alpha_\epsilon u_\epsilon(y)^2} u_\epsilon(y) + \alpha u_\epsilon(y) \right) dy,$$

for $i = 1, 2$. Thus, for any $R > 0$, $x \in B_R(0)$ and $i = 1, 2$, by using (3.2) we have

$$\begin{aligned} |\nabla^i \varphi_\epsilon(x)| &= \left| r_\epsilon^i b_\epsilon \int_{\Omega} \nabla_x^i G(x_\epsilon + r_\epsilon x, y) \left(\frac{1}{\lambda_\epsilon} e^{\alpha_\epsilon u_\epsilon(y)^2} u_\epsilon(y) + \alpha u_\epsilon(y) \right) dy \right| \\ &\leq C b_\epsilon r_\epsilon^i \int_{\Omega} \left(\frac{1}{\lambda_\epsilon} \frac{|u_\epsilon(y)| e^{\alpha_\epsilon u_\epsilon(y)^2}}{|x_\epsilon + r_\epsilon x - y|^i} + \frac{\alpha |u_\epsilon(y)|}{|x_\epsilon + r_\epsilon x - y|^i} \right) dy \\ &\leq C b_\epsilon r_\epsilon^i \left(\int_{B_{2Rr_\epsilon}(x_\epsilon)} \frac{1}{\lambda_\epsilon} \frac{|u_\epsilon(y)| e^{\alpha_\epsilon u_\epsilon(y)^2}}{|x_\epsilon + r_\epsilon x - y|^i} dy + \int_{\Omega \setminus B_{2Rr_\epsilon}(x_\epsilon)} \frac{1}{\lambda_\epsilon} \frac{|u_\epsilon(y)| e^{\alpha_\epsilon u_\epsilon(y)^2}}{|x_\epsilon + r_\epsilon x - y|^i} dy \right. \\ &\quad \left. + \int_{\Omega} \frac{\alpha |u_\epsilon(y)|}{|x_\epsilon + r_\epsilon x - y|^i} dy \right) \\ &\leq C \left(\frac{b_\epsilon}{c_\epsilon} \int_{B_{2R}(0)} \frac{dz}{|x - z|^i} + \frac{1}{R^i} + \alpha b_\epsilon r_\epsilon^i c_\epsilon \int_{\Omega} \frac{dy}{|x_\epsilon + r_\epsilon x - y|^i} \right) \\ &\leq C(R), \end{aligned} \quad (3.9)$$

here we use (3.7) and $b_\epsilon \leq c_\epsilon$.

A straightforward computation shows that φ_ϵ satisfies

$$\Delta^2 \varphi_\epsilon(x) = \frac{b_\epsilon}{c_\epsilon} \psi_\epsilon(x) e^{\alpha_\epsilon \frac{c_\epsilon}{b_\epsilon} (1+\psi_\epsilon(x)) \varphi_\epsilon(x)} + \alpha b_\epsilon c_\epsilon r_\epsilon^4 \psi_\epsilon(x). \quad (3.10)$$

Since $b_\epsilon \leq c_\epsilon$, $\psi_\epsilon \rightarrow 1$ in $C_{\text{loc}}^4(\mathbb{R}^4)$, (3.7), $\varphi_\epsilon \leq 0$ and (3.9), we obtain by applying the standard regularity theory to (3.10) that $\varphi_\epsilon \rightarrow \varphi$ in $C_{\text{loc}}^4(\mathbb{R}^4)$ for some function φ . We

have two following cases.

- *Case 1:* $\tau < \infty$. By letting $\epsilon \rightarrow 0$, then using Lemma 3.1, (3.7) and (3.10) we obtain

$$\Delta^2 \varphi(x) = \frac{1}{\tau} e^{64\pi^2 \tau \varphi(x)}, \quad \varphi(x) \leq \varphi(0) = 0, \quad \int_{\mathbb{R}^4} e^{64\pi^2 \tau \varphi(x)} dx < \infty. \quad (3.11)$$

Indeed, for any $R > 0$, we have $\psi_\epsilon(x) = 1 + o_{\epsilon,R}(1)$ where $o_{\epsilon,R}(1)$ means that

$$\lim_{\epsilon \rightarrow 0} o_{\epsilon,R}(1) = 0 \quad \text{uniformly in } B_R(0).$$

Thus $\psi_\epsilon^2(x) = 1 + o_{\epsilon,R}(1)$ for $x \in B_R(0)$ or equivalently $u_\epsilon(x_\epsilon + r_\epsilon x)^2 = c_\epsilon^2(1 + o_{\epsilon,R}(1))$ for $x \in B_R(0)$. Hence

$$\lambda_\epsilon = \int_{\Omega} u_\epsilon^2 e^{\alpha_\epsilon u_\epsilon^2} dx \geq c_\epsilon^2(1 + o_{\epsilon,R}(1)) \int_{B_{Rr_\epsilon}(x_\epsilon)} e^{\alpha_\epsilon u_\epsilon^2} dx.$$

Applying Fatou's lemma, we have

$$\begin{aligned} \int_{B_R(0)} e^{64\pi^2 \tau \varphi^2(x)} dx &\leq \liminf_{\epsilon \rightarrow 0} \int_{B_R(0)} e^{\alpha_\epsilon \frac{c_\epsilon}{b_\epsilon} (1 + \psi_\epsilon(x)) \varphi_\epsilon(x)} dx \\ &= \liminf_{\epsilon \rightarrow 0} \int_{B_R(0)} e^{\alpha_\epsilon (u_\epsilon(x_\epsilon + r_\epsilon x)^2 - c_\epsilon^2)} dx \\ &= \liminf_{\epsilon \rightarrow 0} r_\epsilon^{-4} \int_{B_{Rr_\epsilon}(x_\epsilon)} e^{\alpha_\epsilon (u_\epsilon(x)^2 - c_\epsilon^2)} dx \\ &= \liminf_{\epsilon \rightarrow 0} \frac{c_\epsilon^2 \int_{B_{Rr_\epsilon}(x_\epsilon)} e^{\alpha_\epsilon u_\epsilon(x)^2} dx}{\lambda_\epsilon} \\ &\leq \liminf_{\epsilon \rightarrow 0} \frac{c_\epsilon^2 \int_{B_{Rr_\epsilon}(x_\epsilon)} e^{\alpha_\epsilon u_\epsilon(x)^2} dx}{c_\epsilon^2(1 + o_{\epsilon,R}(1)) \int_{B_{Rr_\epsilon}(x_\epsilon)} e^{\alpha_\epsilon u_\epsilon(x)^2} dx} \\ &= 1, \end{aligned}$$

for any $R > 0$. Letting $R \rightarrow \infty$ we get $\int_{\mathbb{R}^4} e^{64\pi^2 \tau \varphi} dx < \infty$.

Moreover, we have

$$\Delta \varphi_\epsilon(x) = b_\epsilon r_\epsilon^2 \int_{\Omega} \Delta_x G(x_\epsilon + r_\epsilon x, y) \left(\frac{1}{\lambda_\epsilon} e^{\alpha_\epsilon u_\epsilon(y)^2} u_\epsilon(y) + \alpha u_\epsilon(y) \right) dy.$$

Hence, for any $R > 0$, by Fubini theorem we get

$$\begin{aligned} \int_{B_R(0)} |\Delta \varphi_\epsilon(x)| dx &\leq C b_\epsilon r_\epsilon^2 \int_{\Omega} \frac{1}{\lambda_\epsilon} e^{\alpha_\epsilon u_\epsilon(y)^2} |u_\epsilon(y)| \int_{B_R(0)} \frac{1}{|x_\epsilon + r_\epsilon x - y|^2} dx dy \\ &\quad + C \alpha b_\epsilon r_\epsilon^2 \int_{\Omega} |u_\epsilon(y)| \int_{B_R(0)} \frac{1}{|x_\epsilon + r_\epsilon x - y|^2} dx dy \\ &\leq C' R^2, \end{aligned}$$

with C' independent of R and ϵ . Letting $\epsilon \rightarrow 0$, we obtain

$$\int_{B_R(0)} |\Delta \varphi(x)| dx \leq C' R^2,$$

for any $R > 0$ with C' independent of R . This fact together (3.11) and the results in [27, 43] implies that

$$\varphi(x) = \frac{1}{16\pi^2\tau} \ln \frac{1}{1 + \frac{\pi}{\sqrt{6}}|x|^2}, \quad x \in \mathbb{R}^4.$$

- *Case 2:* $\tau = \infty$. From (3.9) we obtain by letting $\epsilon \rightarrow 0$ that

$$|\Delta \varphi(x)| \leq \frac{C}{R^2},$$

for any $x \in B_R(0)$ and for any $R > 0$ with C independent of R . Let $R \rightarrow \infty$ we get $\Delta \varphi(x) = 0$ for any $x \in \mathbb{R}^4$. Since $\varphi(x) \leq \varphi(0) = 0$ for any $x \in \mathbb{R}^4$, then by Liouville Theorem, we conclude that $\varphi \equiv 0$. \square

We next consider the asymptotic behavior of u_ϵ away from the blow-up point p . We have the following result.

Lemma 3.3. *$b_\epsilon u_\epsilon$ is bounded in $H_0^{2,r}(\Omega)$ for any $1 < r < 2$. In particular, there exists a constant C depending only on Ω , $\lambda_1(\Omega)$ and α_0 such that $\|b_\epsilon u_\epsilon\|_{H_0^{2,r}(\Omega)} \leq C$ uniformly for $\alpha \in [0, \alpha_0]$ with $\alpha_0 < \lambda_1(\Omega)$.*

Proof. Let v_ϵ be the solution of

$$\begin{cases} \Delta^2 v_\epsilon = \frac{1}{\lambda_\epsilon} b_\epsilon u_\epsilon e^{\alpha_\epsilon u_\epsilon^2} & \text{in } \Omega, \\ v_\epsilon = \frac{\partial v_\epsilon}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

By Green representation formula, we have

$$v_\epsilon(x) = \int_{\Omega} G(x, y) \frac{1}{\lambda_\epsilon} b_\epsilon u_\epsilon(y) e^{\alpha_\epsilon u_\epsilon(y)^2} dy$$

and hence for any $i = 1, 2$, it holds

$$|\nabla^i v_\epsilon(x)| \leq C \frac{b_\epsilon}{\lambda_\epsilon} \int_{\Omega} |x - y|^{-i} |u_\epsilon(y)| e^{\alpha_\epsilon u_\epsilon(y)^2} dy = C \int_{\Omega} |x - y|^{-i} \frac{|u_\epsilon(y)| e^{\alpha_\epsilon u_\epsilon(y)^2}}{\int_{\Omega} |u_\epsilon(z)| e^{\alpha_\epsilon u_\epsilon(z)^2} dz} dy.$$

Applying Hölder inequality, we obtain for any $1 < r < 2$ that

$$|\nabla^i v_\epsilon(x)|^r \leq C^r \int_{\Omega} |x - y|^{-ir} \frac{|u_\epsilon(y)| e^{\alpha_\epsilon u_\epsilon(y)^2}}{\int_{\Omega} |u_\epsilon(z)| e^{\alpha_\epsilon u_\epsilon(z)^2} dz} dy.$$

Thus, by Fubini theorem, we have $\|\nabla^i v_\epsilon\|_r \leq C$ for $i = 1, 2$, whence

$$\|v_\epsilon\|_{H_0^{2,r}} \leq C. \quad (3.12)$$

Let $w_\epsilon = b_\epsilon u_\epsilon - v_\epsilon$, then w_ϵ satisfies

$$\begin{cases} \Delta^2 w_\epsilon = \alpha w_\epsilon + \alpha v_\epsilon & \text{in } \Omega, \\ w_\epsilon = \frac{\partial w_\epsilon}{\partial \nu} = 0, & \text{on } \partial\Omega. \end{cases}$$

Using w_ϵ as testing function for this equation, we get

$$\|\Delta w_\epsilon\|_2^2 = \alpha \|w_\epsilon\|_2^2 + \alpha \int_\Omega v_\epsilon w_\epsilon dx \leq \frac{\alpha}{\lambda_1(\Omega)} \|\Delta w_\epsilon\|_2^2 + \frac{\alpha}{\sqrt{\lambda_1(\Omega)}} \|v_\epsilon\|_2 \|\Delta w_\epsilon\|_2.$$

Thus

$$\left(1 - \frac{\alpha}{\lambda_1(\Omega)}\right) \|\Delta w_\epsilon\|_2 \leq \frac{\alpha}{\sqrt{\lambda_1(\Omega)}} \|v_\epsilon\|_2,$$

which together (3.12) and Sobolev inequality yields $\|\Delta w_\epsilon\|_2 \leq C$ with C depends on Ω , $\lambda_1(\Omega)$, and $\alpha_0 < \lambda_1(\Omega)$ such that $0 \leq \alpha \leq \alpha_0$. Hence $\|w_\epsilon\|_{H_0^2(\Omega)} \leq C$ which together (3.12) implies that $b_\epsilon u_\epsilon$ is bounded in $H_0^{2,r}(\Omega)$ for any $1 < r < 2$. \square

We proceed by showing that $b_\epsilon u_\epsilon$ converges to some Green function.

Lemma 3.4. *It holds $b_\epsilon u_\epsilon \rightharpoonup G_\alpha(\cdot, p)$ in $H_0^{2,r}(\Omega)$ for any $1 < r < 2$ with*

$$\begin{cases} \Delta^2 G_\alpha(\cdot, p) = \sigma \delta_p + \alpha G_\alpha(\cdot, p) & \text{in } \Omega, \\ G_\alpha(\cdot, p) = \frac{\partial G_\alpha(\cdot, p)}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.13)$$

Furthermore, $b_\epsilon u_\epsilon \rightarrow G_\alpha(\cdot, p)$ in $C_{\text{loc}}^4(\overline{\Omega} \setminus \{p\})$. Also, we have

$$G_\alpha(x, p) = -\frac{\sigma}{8\pi^2} \ln |x - p| + A_p + \psi(x), \quad (3.14)$$

where A_p is constant depending on p and α , $\psi \in C^3(\overline{\Omega})$ and $\psi(p) = 0$.

Proof. By Lemma 3.3, there exists a function $G_\alpha(\cdot, p) \in H_0^{2,s}(\Omega)$ such that $b_\epsilon u_\epsilon \rightharpoonup G_\alpha(\cdot, p)$ weakly in $H_0^{2,s}(\Omega)$ for any $1 < s < 2$. For any $r > 0$ such that $B_r(p) \subset \Omega$, by (3.5) we have $e^{\alpha_\epsilon u_\epsilon^2}$ is bounded in $L^s(\Omega \setminus B_r(p))$ for any $s > 1$ (this is based on Adams inequality and cut-off function argument). Hence, by the standard regularity theory we obtain $b_\epsilon u_\epsilon \rightarrow G_\alpha(\cdot, p)$ in $C_{\text{loc}}^4(\overline{\Omega} \setminus \{p\})$. Notice that $b_\epsilon u_\epsilon$ satisfies

$$\begin{cases} \Delta^2(b_\epsilon u_\epsilon) = \frac{1}{\lambda_\epsilon} b_\epsilon u_\epsilon e^{\alpha_\epsilon u_\epsilon^2} + \alpha b_\epsilon u_\epsilon & \text{in } \Omega, \\ b_\epsilon u_\epsilon = \frac{\partial(b_\epsilon u_\epsilon)}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.15)$$

For any $\phi \in C^\infty(\overline{\Omega})$ we have

$$\begin{aligned} \int_\Omega \phi \left(\frac{1}{\lambda_\epsilon} b_\epsilon u_\epsilon e^{\alpha_\epsilon u_\epsilon^2} + \alpha b_\epsilon u_\epsilon \right) dx &= \int_\Omega (\phi - \phi(p)) \frac{1}{\lambda_\epsilon} b_\epsilon u_\epsilon e^{\alpha_\epsilon u_\epsilon^2} dx \\ &\quad + \phi(p) \int_\Omega \frac{1}{\lambda_\epsilon} b_\epsilon u_\epsilon e^{\alpha_\epsilon u_\epsilon^2} dx + \alpha \int_\Omega b_\epsilon u_\epsilon \phi dx. \end{aligned} \quad (3.16)$$

Note that

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega} b_{\epsilon} u_{\epsilon} \phi dx = \int_{\Omega} G_{\alpha}(x, p) \phi(x) dx, \quad (3.17)$$

and

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega} \frac{1}{\lambda_{\epsilon}} b_{\epsilon} u_{\epsilon} e^{\alpha_{\epsilon} u_{\epsilon}^2} dx = \lim_{\epsilon \rightarrow 0} \frac{\int_{\Omega} u_{\epsilon}(x) e^{\alpha_{\epsilon} u_{\epsilon}(x)^2} dx}{\int_{\Omega} |u_{\epsilon}(x)| e^{\alpha_{\epsilon} u_{\epsilon}(x)^2} dx} = \sigma. \quad (3.18)$$

We will show that

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega} (\phi - \phi(p)) \frac{1}{\lambda_{\epsilon}} b_{\epsilon} u_{\epsilon} e^{\alpha_{\epsilon} u_{\epsilon}^2} dx = 0. \quad (3.19)$$

Indeed, by Lebesgue dominated convergence theorem, we get that

$$\lim_{\epsilon \rightarrow 0} \int_{\{|u_{\epsilon}| \leq 1\}} e^{\alpha_{\epsilon} u_{\epsilon}^2} dx = |\Omega|.$$

This limit and (2.5) imply

$$\begin{aligned} \liminf_{\epsilon \rightarrow 0} \int_{\Omega} |u_{\epsilon}| e^{\alpha_{\epsilon} u_{\epsilon}^2} dx &\geq \liminf_{\epsilon \rightarrow 0} \int_{\{|u_{\epsilon}| \geq 1\}} e^{\alpha_{\epsilon} u_{\epsilon}^2} dx \\ &= \liminf_{\epsilon \rightarrow 0} \left(C_{\epsilon} - \int_{\{|u_{\epsilon}| \leq 1\}} e^{\alpha_{\epsilon} u_{\epsilon}^2} dx \right) \\ &= \sup_{u \in H_0^2(\Omega), \|u\|_{2,\alpha} = 1} \int_{\Omega} e^{32\pi^2 u^2} dx - |\Omega| \\ &> 0, \end{aligned}$$

hence $b_{\epsilon}/\lambda_{\epsilon}$ is bounded. For any $r > 0$ with $B_r(p) \subset \Omega$, we know that $e^{\alpha_{\epsilon} u_{\epsilon}^2}$ is bounded in $L^s(\Omega \setminus B_r(p))$ for some $s > 1$ and $u_{\epsilon} \rightarrow 0$ in $L^t(\Omega)$ for any $t > 1$, hence

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega \setminus B_r(p)} (\phi - \phi(p)) \frac{b_{\epsilon}}{\lambda_{\epsilon}} u_{\epsilon} e^{\alpha_{\epsilon} u_{\epsilon}^2} dx = 0. \quad (3.20)$$

In the other hand

$$\left| \int_{B_r(p)} (\phi - \phi(p)) \frac{b_{\epsilon}}{\lambda_{\epsilon}} u_{\epsilon} e^{\alpha_{\epsilon} u_{\epsilon}^2} dx \right| \leq \sup_{x \in B_r(p)} |\phi(x) - \phi(p)| \frac{b_{\epsilon}}{\lambda_{\epsilon}} \int_{\Omega} |u_{\epsilon}| e^{\alpha_{\epsilon} u_{\epsilon}^2} dx = \sup_{x \in B_r(p)} |\phi(x) - \phi(p)|.$$

Thus

$$\lim_{r \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \int_{B_r(p)} (\phi - \phi(p)) \frac{b_{\epsilon}}{\lambda_{\epsilon}} u_{\epsilon} e^{\alpha_{\epsilon} u_{\epsilon}^2} dx = 0. \quad (3.21)$$

(3.19) follows from (3.20) and (3.21).

Plugging (3.17), (3.18) and (3.19) into (3.16) we obtain

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega} \phi \left(\frac{1}{\lambda_{\epsilon}} b_{\epsilon} u_{\epsilon} e^{\alpha_{\epsilon} u_{\epsilon}^2} + \alpha b_{\epsilon} u_{\epsilon} \right) dx = \sigma \phi(p) + \alpha \int_{\Omega} G_{\alpha}(x, p) \phi(x) dx,$$

for any $\phi \in C^\infty(\overline{\Omega})$, hence

$$\Delta^2 G_\alpha(\cdot, p) = \sigma \delta_p + \alpha G_\alpha(\cdot, p) \quad \text{in } \Omega.$$

The last conclusion was proved in the proof of Lemma 4.4 in [30]. Let us recall it here for convenience of reader. Fix $r > 0$ such that $B_{2r}(p) \subset \Omega$ and consider the cut-off function $\phi \in C_0^\infty(B_{2r}(p))$ such that $\phi \equiv 1$ in $B_r(p)$. Let

$$g(x) = G_\alpha(x, p) + \frac{\sigma}{8\pi^2} \eta(x) \ln |x - p|.$$

Then we have

$$\begin{cases} \Delta^2 g = f & \text{in } \Omega, \\ g = \frac{\partial g}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

with

$$\begin{aligned} f(x) = & -\frac{\sigma^2}{8\pi^2} \left(\Delta^2 \phi(x) \ln |x - p| + 2 \nabla \Delta \phi(x) \nabla \ln |x - p| + 2 \Delta \phi(x) \Delta \ln |x - p| \right. \\ & \left. + 2 \Delta (\nabla \phi(x) \nabla \ln |x - p|) + 2 \nabla \phi(x) \nabla \Delta \ln |x - p| \right) + \alpha G_\alpha(x, p). \end{aligned}$$

Lemma 3.3 and Sobolev inequality implies that $f \in L^s(\Omega)$ for any $s > 1$. By the standard regularity theory, we have $g \in C^3(\overline{\Omega})$. Let $A_p = g(p)$ and

$$\psi(x) = g(x) - g(p) + \frac{\sigma}{8\pi^2} (1 - \phi(x)) \ln |x - p|,$$

we obtain the desired result. \square

We continue by using Pohozaev type identity to find an upper bound of $\int_\Omega e^{\alpha_\epsilon u_\epsilon^2} dx$. The following Pohozaev type identity is very useful in our analysis below.

Lemma 3.5. *Assume $\Omega' \subset \mathbb{R}^4$ is a smooth bounded domain. Let $u \in C^4(\overline{\Omega'})$ be a solution of $\Delta^2 u = f(u)$ in Ω' . Then we have for any $y \in \mathbb{R}^4$*

$$\begin{aligned} 4 \int_{\Omega'} F(u) dx = & \int_{\partial\Omega'} \langle x - y, \nu \rangle F(u) d\omega + \frac{1}{2} \int_{\partial\Omega'} v^2 \langle x - y, \nu \rangle d\omega + 2 \int_{\partial\Omega'} \frac{\partial u}{\partial \nu} v d\omega \\ & + \int_{\partial\Omega'} \left(\frac{\partial v}{\partial \nu} \langle x - y, \nabla u \rangle + \frac{\partial u}{\partial \nu} \langle x - y, \nabla v \rangle - \langle \nabla u, \nabla v \rangle \langle x - y, \nu \rangle \right) d\omega, \end{aligned}$$

where $F(u) = \int_0^u f(s) ds$, $v = -\Delta u$ and ν is the normal outward derivative of x on $\partial\Omega'$.

The proof of this Pohozaev type identity can be found in [33, 36]. In the sequel, we will apply it for $\Omega' = B_r(x_\epsilon)$, $y = x_\epsilon$, $u = u_\epsilon$ and $f(u) = \frac{1}{\lambda_\epsilon} u e^{\alpha_\epsilon u^2} + \alpha u$. Noting that $v = -\Delta u_\epsilon$ and $F(u) = \frac{1}{2\alpha_\epsilon \lambda_\epsilon} e^{\alpha_\epsilon u^2} + \frac{\alpha}{2} u^2$. By Lemma 3.5, we have

$$\begin{aligned} & \int_{B_r(x_\epsilon)} e^{\alpha_\epsilon u_\epsilon^2} dx \\ &= -\frac{\alpha \alpha_\epsilon \lambda_\epsilon}{b_\epsilon^2} \int_{B_r(x_\epsilon)} (b_\epsilon u_\epsilon)^2 dx + \frac{r}{4} \int_{\partial B_r(x_\epsilon)} e^{\alpha_\epsilon u_\epsilon^2} d\omega + \frac{\alpha \alpha_\epsilon \lambda_\epsilon r}{b_\epsilon^2} \frac{1}{4} \int_{\partial B_r(x_\epsilon)} (b_\epsilon u_\epsilon)^2 d\omega \\ & \quad + \frac{\alpha_\epsilon \lambda_\epsilon}{4b_\epsilon^2} r \int_{\partial B_r(x_\epsilon)} |\Delta(b_\epsilon u_\epsilon)|^2 d\omega - \frac{\alpha_\epsilon \lambda_\epsilon}{b_\epsilon^2} \int_{\partial B_r(x_\epsilon)} \frac{\partial(b_\epsilon u_\epsilon)}{\partial \nu} \Delta(b_\epsilon u_\epsilon) d\omega \\ & \quad - \frac{\alpha_\epsilon \lambda_\epsilon}{2b_\epsilon^2} r \int_{\partial B_r(x_\epsilon)} \left(2 \frac{\partial \Delta(b_\epsilon u_\epsilon)}{\partial \nu} \frac{\partial(b_\epsilon u_\epsilon)}{\partial \nu} - \langle \nabla \Delta(b_\epsilon u_\epsilon), \nabla(b_\epsilon u_\epsilon) \rangle \right) d\omega. \end{aligned} \quad (3.22)$$

Using the representation of G_α in Lemma 3.4 and $x_\epsilon \rightarrow p$, we have

$$\begin{aligned} & \int_{B_r(x_\epsilon)} (b_\epsilon u_\epsilon)^2 dx = \int_{B_r(p)} G_\alpha(x, p)^2 dx + o_{\epsilon, r}(1) = o_r(1) + o_{\epsilon, r}(1), \\ & r \int_{\partial B_r(x_\epsilon)} e^{\alpha_\epsilon u_\epsilon} d\omega = o_r(1) + o_{\epsilon, r}(1), \quad r \int_{\partial B_r(x_\epsilon)} (b_\epsilon u_\epsilon)^2 d\omega = o_r(1) + o_{\epsilon, r}(1), \\ & r \int_{\partial B_r(x_\epsilon)} |\Delta(b_\epsilon u_\epsilon)|^2 d\omega = r \int_{\partial B_r(p)} |\Delta G_\alpha(x, p)|^2 d\omega + o_{\epsilon, r}(1) = \frac{\sigma^2}{8\pi^2} + o_r(1) + o_{\epsilon, r}(1), \\ & \int_{\partial B_r(x_\epsilon)} \frac{\partial(b_\epsilon u_\epsilon)}{\partial \nu} \Delta(b_\epsilon u_\epsilon) d\omega = \int_{\partial B_r(p)} \frac{\partial G_\alpha(x, p)}{\partial \nu} \Delta G_\alpha(x, p) d\omega + o_{\epsilon, r}(1) = \frac{\sigma^2}{16\pi^2} + o_r(1) + o_{\epsilon, r}(1), \\ & r \int_{\partial B_r(x_\epsilon)} \frac{\partial \Delta(b_\epsilon u_\epsilon)}{\partial \nu} \frac{\partial(b_\epsilon u_\epsilon)}{\partial \nu} d\omega = r \int_{\partial B_r(p)} \frac{\partial \Delta(G_\alpha)}{\partial \nu} \frac{\partial(G_\alpha)}{\partial \nu} d\omega + o_{\epsilon, r}(1) = -\frac{\sigma^2}{8\pi^2} + o_r(1) + o_{\epsilon, r}(1), \end{aligned}$$

and

$$r \int_{\partial B_r(x_\epsilon)} \langle \nabla \Delta(b_\epsilon u_\epsilon), \nabla(b_\epsilon u_\epsilon) \rangle d\omega = r \int_{\partial B_r(p)} \langle \nabla \Delta G_\alpha, \nabla G_\alpha \rangle d\omega + o_{\epsilon, r}(1) = -\frac{\sigma^2}{8\pi^2} + o_r(1) + o_{\epsilon, r}(1),$$

where $o_{\epsilon, r}(1)$ and $o_r(1)$ mean that $\lim_{\epsilon \rightarrow 0} o_{\epsilon, r}(1) = 0$ when r is fixed, and $\lim_{r \rightarrow 0} o_r(1) = 0$ respectively. Hence, we get

$$\int_{B_r(x_\epsilon)} e^{\alpha_\epsilon u_\epsilon^2} dx = \frac{\lambda_\epsilon}{b_\epsilon^2} (\sigma^2 + o_r(1) + o_{\epsilon, r}(1)) + o_r(1) + o_{\epsilon, r}(1). \quad (3.23)$$

We claim that $\sigma^2 > 0$. Indeed, if this is not true, then $\sigma^2 = 0$, and we have

$$\int_{B_r(x_\epsilon)} e^{\alpha_\epsilon u_\epsilon^2} dx = \frac{\lambda_\epsilon}{b_\epsilon^2} (o_r(1) + o_{\epsilon, r}(1)) + o_r(1) + o_{\epsilon, r}(1).$$

Since $|\Delta u_\epsilon|^2 dx \rightharpoonup \delta_p$ in the measure sense, then for any $r > 0$ with $B_r(p) \subset \Omega$, by using Adams inequality and cut-off function argument, we have

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega \setminus B_r(p)} e^{\alpha_\epsilon u_\epsilon^2} dx = |\Omega| - |B_r(p)|. \quad (3.24)$$

Fix a $r_0 > 0$ such that $B_{2r_0}(p) \subset \Omega$ and $|o_r(1)| \leq 1/4$ for any $r \leq 2r_0$. Choosing $\epsilon_0 > 0$ such that $|x_\epsilon - p| < r_0$ and $|o_{\epsilon, 2r_0}(1)| \leq 1/4$ for any $\epsilon \leq \epsilon_0$. Thus $B_{r_0}(p) \subset B_{2r_0}(x_\epsilon)$, and hence

$$\limsup_{\epsilon \rightarrow 0} \int_{\Omega \setminus B_{2r_0}(x_\epsilon)} e^{\alpha_\epsilon u_\epsilon^2} dx \leq |\Omega| - |B_{r_0}(p)| \leq |\Omega|.$$

By Hölder inequality, we have

$$\frac{\lambda_\epsilon}{b_\epsilon^2} = \frac{\left(\int_{\Omega} |u_\epsilon| e^{\alpha_\epsilon u_\epsilon^2} dx \right)^2}{\int_{\Omega} u_\epsilon^2 e^{\alpha_\epsilon u_\epsilon^2} dx} \leq \int_{\Omega} e^{\alpha_\epsilon u_\epsilon^2} dx \leq \frac{1}{2} \frac{\lambda_\epsilon}{b_\epsilon^2} + \frac{1}{2} + \int_{\Omega \setminus B_{2r_0}(x_\epsilon)} e^{\alpha_\epsilon u_\epsilon^2} dx.$$

Thus

$$\limsup_{\epsilon \rightarrow 0} \frac{\lambda_\epsilon}{b_\epsilon^2} \leq 1 + 2|\Omega| < \infty.$$

This together the estimates above and $\sigma^2 = 0$ implies

$$\lim_{r \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{B_r(x_\epsilon)} e^{\alpha_\epsilon u_\epsilon^2} dx = 0.$$

For any $r > 0$ such that $B_{2r}(p) \subset \Omega$, we then have $B_{r/2}(p) \subset B_r(x_\epsilon) \subset B_{2p}(p)$ for sufficiently small $\epsilon > 0$. Thus by (3.24), it holds

$$\lim_{r \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{\Omega \setminus B_r(x_\epsilon)} e^{\alpha_\epsilon u_\epsilon^2} dx = |\Omega|. \quad (3.25)$$

Finally, we get

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega} e^{\alpha_\epsilon u_\epsilon^2} dx \leq |\Omega|,$$

which is impossible. Then we must have $\sigma^2 > 0$. This together (3.23) and (3.25) yields

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega} e^{\alpha_\epsilon u_\epsilon^2} dx = |\Omega| + \sigma^2 \lim_{\epsilon \rightarrow 0} \frac{\lambda_\epsilon}{b_\epsilon^2}. \quad (3.26)$$

We can further locate σ as follows.

Lemma 3.6. *It holds $\sigma = 1$.*

Proof. We know from Lemma 3.4 that $b_\epsilon u_\epsilon \rightarrow G_\alpha(\cdot, p)$ in $C_{\text{loc}}^4(\overline{\Omega} \setminus \{p\})$ with

$$G_\alpha(x, p) = -\frac{\sigma}{8\pi^2} \ln|x-p| + A_p + \psi(x), \quad \psi \in C^3(\overline{\Omega}), \psi(p) = 0.$$

We also know that $\sigma \neq 0$. Suppose $\sigma < 0$, then $G_\alpha(\cdot, p) \leq -C$ in $B_r(p)$ for some $r > 0$ and $C > 0$. Hence $u_\epsilon < 0$ in $B_r(p) \setminus \{p\}$ for ϵ small enough. In the other hand, by Hölder inequality, we have

$$b_\epsilon \geq \frac{\int_\Omega |u_\epsilon| e^{\alpha_\epsilon u_\epsilon^2} dx}{\int_\Omega e^{\alpha_\epsilon u_\epsilon^2} dx} \geq 1 - \frac{\int_{\{|u_\epsilon| \leq 1\}} e^{\alpha_\epsilon u_\epsilon^2} dx}{\int_\Omega e^{\alpha_\epsilon u_\epsilon^2} dx},$$

thus

$$\liminf_{\epsilon \rightarrow 0} b_\epsilon \geq 1 - \frac{|\Omega|}{\sup_{u \in H_0^2(\Omega), \|u\|_{2,\alpha}=1} \int_\Omega e^{32\pi^2 u^2} dx} > 0.$$

Lemma 3.2 implies that $u_\epsilon > 0$ on $B_{Rr_\epsilon}(x_\epsilon)$ for any fixed $R > 0$ provided that $\epsilon > 0$ is small enough (since $c_\epsilon \rightarrow \infty$). However when ϵ is small enough, we then have $B_{Rr_\epsilon}(x_\epsilon) \subset B_r(p)$. We thus get a contradiction on the sign of u_ϵ , hence $\sigma > 0$. Whence, $G_\alpha(\cdot, p) \geq C > 0$ in $B_r(p) \setminus \{p\}$ for some $r > 0$ and $C > 0$, hence $u_\epsilon > 0$ in $B_r(p) \setminus \{p\}$ for sufficiently small $\epsilon > 0$, and then we have

$$\int_{B_r(p)} |u_\epsilon| e^{\alpha_\epsilon u_\epsilon^2} dx = \int_{B_r(p)} u_\epsilon e^{\alpha_\epsilon u_\epsilon^2} dx.$$

Since $|\Delta u_\epsilon|^2 dx \rightharpoonup \delta_p$ in the measure sense, and $u_\epsilon \rightarrow 0$ in $L^s(\Omega)$ for any $s > 1$, then by using Adams inequality and cut-off function argument, we can show that

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega \setminus B_r(p)} |u_\epsilon| e^{\alpha_\epsilon u_\epsilon^2} dx = 0.$$

Obviously,

$$\int_\Omega |u_\epsilon| e^{\alpha_\epsilon u_\epsilon^2} dx \geq \int_{\{|u_\epsilon| \geq 1\}} e^{\alpha_\epsilon u_\epsilon^2} dx = \int_\Omega e^{\alpha_\epsilon u_\epsilon^2} dx - \int_{\{|u_\epsilon| \leq 1\}} e^{\alpha_\epsilon u_\epsilon^2} dx$$

hence by (2.5) and Lebesgue dominated convergence theorem we have

$$\liminf_{\epsilon \rightarrow 0} \int_\Omega |u_\epsilon| e^{\alpha_\epsilon u_\epsilon^2} dx \geq \sup_{u \in H_0^2(\Omega), \|u\|_{2,\alpha}=1} \int_\Omega e^{32\pi^2 u^2} dx - |\Omega| > 0.$$

Since

$$|\sigma - 1| = \left| \lim_{\epsilon \rightarrow 0} \frac{\int_\Omega u_\epsilon e^{\alpha_\epsilon u_\epsilon^2} dx}{\int_\Omega |u_\epsilon| e^{\alpha_\epsilon u_\epsilon^2} dx} - 1 \right| \leq 2 \lim_{\epsilon \rightarrow 0} \frac{\int_{\Omega \setminus B_r(p)} |u_\epsilon| e^{\alpha_\epsilon u_\epsilon^2} dx}{\int_\Omega |u_\epsilon| e^{\alpha_\epsilon u_\epsilon^2} dx} = 0.$$

Hence $\sigma = 1$. □

To summarize, we have the following result.

Lemma 3.7. $b_\epsilon u_\epsilon \rightharpoonup G_\alpha(\cdot, p)$ weakly in $H_0^{2,r}(\Omega)$ for any $1 < r < 2$ with

$$\begin{cases} \Delta^2 G_\alpha(\cdot, p) = \delta_p + \alpha G_\alpha(\cdot, p) & \text{in } \Omega \\ G_\alpha(\cdot, p) = \frac{\partial G_\alpha(\cdot, p)}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Furthermore, $b_\epsilon u_\epsilon \rightarrow G_\alpha(\cdot, p)$ in $C_{\text{loc}}^4(\overline{\Omega} \setminus \{p\})$. Also we have

$$G_\alpha(x, p) = -\frac{1}{8\pi^2} \ln |x - p| + A_p + \psi(x),$$

where A_p is constant depending on p and α , $\psi \in C^3(\overline{\Omega})$, with $\psi(p) = 0$.

4 Capacity estimates

We follow the argument in [30]. Notice that in this section, we still assume that u_ϵ blows up and the blow-up point $p \in \Omega$. We use capacity estimates to calculate the limit of $\lambda_\epsilon/b_\epsilon^2$ to estimate from above the supremum of the functional $\int_\Omega e^{32\pi^2 u^2} dx$ over functions $u \in H_0^2(\Omega)$ with $\|u\|_{2,\alpha} = 1$ under the assumption that u_ϵ blows up. The technique of using capacity estimate applied to this kind of problems was discovery by Li [22] in dealing with Moser–Trudinger inequality of first order derivatives.

Let u_ϵ^* be the function constructed by Lu and Yang in section §5 in [30]. The main properties of this function are that $u_\epsilon^* \in H^2(B_\delta(x_\epsilon) \setminus B_{Rr_\epsilon}(x_\epsilon))$ and satisfies the boundary conditions

$$\begin{cases} u_\epsilon^*(x) = \frac{1}{b_\epsilon} \left(\frac{1}{8\pi^2} \ln \frac{1}{\delta} + A_p \right) & \text{on } \partial B_\delta(x_\epsilon), \\ u_\epsilon^*(x) = c_\epsilon + \frac{1}{b_\epsilon} \varphi \left(\frac{x - x_\epsilon}{r_\epsilon} \right) & \text{on } \partial B_{Rr_\epsilon}(x_\epsilon), \\ \frac{\partial u_\epsilon^*}{\partial \nu} = -\frac{1}{8\pi^2 \delta b_\epsilon} & \text{on } \partial B_\delta(x_\epsilon), \\ \frac{\partial u_\epsilon^*}{\partial \nu} = \frac{1}{b_\epsilon r_\epsilon} \frac{\partial \varphi}{\partial \nu} \left(\frac{x - x_\epsilon}{r_\epsilon} \right) & \text{on } \partial B_{Rr_\epsilon}(x_\epsilon), \end{cases} \quad (4.1)$$

and energy identity

$$\int_{B_\delta(x_\epsilon) \setminus B_{Rr_\epsilon}(x_\epsilon)} |\Delta u_\epsilon^*|^2 dx = \int_{B_\delta(x_\epsilon) \setminus B_{Rr_\epsilon}(x_\epsilon)} |\Delta u_\epsilon|^2 dx + \frac{o(1)}{b_\epsilon^2}. \quad (4.2)$$

Now we start to derive the capacity estimates. Consider the variational problem

$$i_{\delta,R,\epsilon} = \inf \int_{B_\delta(x_\epsilon) \setminus B_{Rr_\epsilon}(x_\epsilon)} |\Delta u|^2 dx$$

where infimum takes all over functions belonging to $H^2(B_\delta(x_\epsilon) \setminus B_{Rr_\epsilon}(x_\epsilon))$ with the same boundary conditions as u_ϵ^* . It is well known (see [24, 26]) that this infimum is attained by a bi-harmonic function \mathcal{T} which is defined in the annular domain $B_\delta(x_\epsilon) \setminus B_{Rr_\epsilon}(x_\epsilon)$ with the same boundary condition as u_ϵ^* . The explicit form of \mathcal{T} is given by

$$\mathcal{T}(x) = \mathcal{A} \ln |x - x_\epsilon| + \mathcal{B} |x - x_\epsilon|^2 + \mathcal{C} |x - x_\epsilon|^{-2} + \mathcal{D},$$

with the explicit values of \mathcal{A}, \mathcal{B} was given in [30] (section §5) by solving a linear system. Hence

$$i_{\delta, R, \epsilon} = 8\pi^2 \mathcal{A}^2 \ln \frac{\delta}{Rr_\epsilon} + 32\pi^2 \mathcal{A}\mathcal{B}(\delta^2 - R^2 r_\epsilon^2) + 32\pi^2 \mathcal{B}^2(\delta^4 - R^4 r_\epsilon^4). \quad (4.3)$$

By the same proof of Lemma 5.1 in [30], we conclude that

$$\lim_{\epsilon \rightarrow 0} \frac{1}{c_\epsilon^2} \ln \frac{\lambda_\epsilon}{c_\epsilon^2} = 0. \quad (4.4)$$

From the definition of r_ϵ , we have

$$\ln \frac{Rr_\epsilon}{\delta} = \ln \frac{R}{\delta} + \frac{\ln \frac{\lambda_\epsilon}{c_\epsilon^2} - \alpha_\epsilon c_\epsilon^2}{4}. \quad (4.5)$$

According to the argument in [30] with the help of (4.4) and (4.5) and using the explicit values of \mathcal{A} and \mathcal{B} , we obtain

$$\begin{aligned} 8\pi^2 \mathcal{A}^2 \ln \frac{\delta}{Rr_\epsilon} &= \frac{32\pi^2}{\alpha_\epsilon} \left(1 + \frac{2\varphi(R) + R\varphi'(R) + \frac{1}{4\pi^2} \ln \delta - 2A_p - \frac{1}{8\pi^2}}{b_\epsilon c_\epsilon} \right. \\ &\quad \left. + \frac{\ln \frac{\lambda_\epsilon}{c_\epsilon^2} + 8 + 4 \ln \frac{R}{r}}{\alpha_\epsilon c_\epsilon^2} + O\left(\frac{1}{c_\epsilon^4} \ln^2 \frac{\lambda_\epsilon}{c_\epsilon^2}\right) + o\left(\frac{1}{b_\epsilon c_\epsilon}\right) \right) \end{aligned} \quad (4.6)$$

and

$$32\pi^2 \mathcal{A}\mathcal{B}(\delta^2 - R^2 r_\epsilon^2) = O\left(\frac{1}{b_\epsilon c_\epsilon}\right), \quad 32\pi^2 \mathcal{B}^2(\delta^4 - R^4 r_\epsilon^4) = O\left(\frac{1}{b_\epsilon^2}\right). \quad (4.7)$$

Remark that (4.6) is exactly the formula (5.12) in [30] with a mistake on the coefficient of $R\varphi'(R)$. We correct this mistake in (4.6). From (4.2) and definition of $i_{\delta, R, \epsilon}$ we have

$$\begin{aligned} i_{\delta, R, \epsilon} &\leq \int_{B_\delta(x_\epsilon) \setminus B_{Rr_\epsilon}(x_\epsilon)} |\Delta u_\epsilon|^2 dx + \frac{o(1)}{b_\epsilon^2} \\ &= 1 + \alpha \|u_\epsilon\|_2^2 - \int_{\Omega \setminus B_\delta(x_\epsilon)} |\Delta u_\epsilon|^2 dx - \int_{B_{Rr_\epsilon}(x_\epsilon)} |\Delta u_\epsilon|^2 dx + \frac{o(1)}{b_\epsilon^2} \\ &= 1 - \frac{1}{b_\epsilon^2} \left(\int_{\Omega \setminus B_\delta(p)} |\Delta G_\alpha|^2 dx + \int_{B_{Rr_\epsilon}(0)} |\Delta \varphi|^2 dx - \alpha \|G_\alpha\|_2^2 \right) + \frac{o(1)}{b_\epsilon^2}, \end{aligned} \quad (4.8)$$

here we use Lemma 3.2 and Lemma 3.4. By integration by parts, we have

$$\int_{\Omega \setminus B_\delta(p)} |\Delta G_\alpha|^2 dx = -\frac{1}{16\pi^2} - \frac{1}{8\pi^2} \ln \delta + A_p + \alpha \|G_\alpha\|_2^2 + O(\delta \ln \delta). \quad (4.9)$$

(4.8) together (4.9) gives

$$i_{\delta, R, \epsilon} \leq 1 - \frac{1}{b_\epsilon^2} \left(\int_{B_{Rr_\epsilon}(0)} |\Delta \varphi|^2 dx - \frac{1}{16\pi^2} - \frac{1}{8\pi^2} \ln \delta + A_p + O(\delta \ln \delta) \right) + \frac{o(1)}{b_\epsilon^2}. \quad (4.10)$$

Plugging (4.3), (4.6) and (4.7) into (4.10) and using the fact $32\pi^2/\alpha_\epsilon > 1$, we obtain

$$\begin{aligned} & \frac{32\pi^2}{\alpha_\epsilon} \left(\frac{2\varphi(R) + R\varphi'(R) + \frac{1}{4\pi^2} \ln \delta - 2A_p - \frac{1}{8\pi^2}}{b_\epsilon c_\epsilon} + \frac{\ln \frac{\lambda_\epsilon}{c_\epsilon^2} + 8 + 4 \ln \frac{R}{r}}{\alpha_\epsilon c_\epsilon^2} \right) \\ & \quad + O\left(\frac{1}{c_\epsilon^4} \ln^2 \frac{\lambda_\epsilon}{c_\epsilon^2}\right) + O\left(\frac{1}{b_\epsilon c_\epsilon}\right) \\ & \leq -\frac{1}{b_\epsilon^2} \left(\int_{B_r(0)} |\Delta \varphi|^2 dx - \frac{1}{16\pi^2} - \frac{1}{8\pi^2} \ln \delta + A_p + O(\delta \ln \delta) \right) + \frac{o(1)}{b_\epsilon^2}. \end{aligned} \quad (4.11)$$

Multiplying both sides of (4.11) by $\alpha_\epsilon c_\epsilon^2$, using the fact $b_\epsilon \leq c_\epsilon$ and making a simple calculation, we obtain

$$\begin{aligned} \left[\frac{32\pi^2}{\alpha_\epsilon} + O\left(\frac{1}{c_\epsilon^2} \ln \frac{\lambda_\epsilon}{c_\epsilon^2}\right) \right] \ln \frac{\lambda_\epsilon}{c_\epsilon^2} & \leq -\frac{\alpha_\epsilon c_\epsilon^2}{b_\epsilon^2} \left(\int_{B_R(0)} |\Delta \varphi|^2 dx - \frac{1}{8\pi^2} \ln \delta \right) - \frac{32\pi^2}{\alpha_\epsilon} 4 \ln \frac{R}{\delta} \\ & \quad - 32\pi^2 \frac{c_\epsilon}{b_\epsilon} \left(2\varphi(R) + R\varphi'(R) + \frac{1}{4\pi^2} \ln \delta \right) + O\left(\frac{c_\epsilon^2}{b_\epsilon^2}\right). \end{aligned}$$

Notice that

$$\ln \frac{\lambda_\epsilon}{c_\epsilon^2} = \ln \frac{\lambda_\epsilon}{b_\epsilon^2} + \ln \frac{b_\epsilon^2}{c_\epsilon^2}, \quad \frac{32\pi^2}{\alpha_\epsilon} = 1 + O(\epsilon).$$

These equalities together (4.4) and the previous inequality imply

$$\begin{aligned} \ln \frac{\lambda_\epsilon}{b_\epsilon^2} & \leq -(1 + o(1)) \frac{\alpha_\epsilon c_\epsilon^2}{b_\epsilon^2} \left(\int_{B_R(0)} |\Delta \varphi|^2 dx - \frac{1}{8\pi^2} \ln \delta \right) - (4 + o(1)) \ln \frac{R}{\delta} \\ & \quad - (1 + o(1)) \frac{\alpha_\epsilon c_\epsilon}{b_\epsilon} \left(2\varphi(R) + R\varphi'(R) + \frac{1}{4\pi^2} \ln \delta \right) - (1 + o(1)) \ln \frac{c_\epsilon^2}{b_\epsilon^2} + O\left(\frac{c_\epsilon^2}{b_\epsilon^2}\right). \end{aligned} \quad (4.12)$$

Notice that by (2.5) and (3.26) we have $\lim_{\epsilon \rightarrow 0} \lambda_\epsilon / b_\epsilon^2 > 0$, hence

$$\ln \frac{\lambda_\epsilon}{b_\epsilon^2} \geq -C_0,$$

for some $C_0 > 0$. If $\tau = \lim_{\epsilon \rightarrow 0} \frac{c_\epsilon}{b_\epsilon} = \infty$ then $\varphi \equiv 0$ by Lemma 3.2 which shows that

$$\ln \frac{\lambda_\epsilon}{b_\epsilon^2} \leq (4 + o(1)) \frac{c_\epsilon^2}{b_\epsilon^2} \ln \delta - (8 + o(1)) \frac{c_\epsilon}{b_\epsilon} \ln \delta - (4 + o(1)) \ln \frac{R}{\delta} + O\left(\frac{c_\epsilon^2}{b_\epsilon^2}\right).$$

Hence for a fixed $R > 0$, by choosing $\delta > 0$ sufficiently small, we have

$$-C_0 \leq \ln \frac{\lambda_\epsilon}{b_\epsilon^2} \leq 2 \frac{c_\epsilon^2}{b_\epsilon^2} \ln \delta - (8 + o(1)) \frac{c_\epsilon}{b_\epsilon} \ln \delta - (4 + o(1)) \ln \frac{R}{\delta},$$

which is impossible since the right hand side tends to $-\infty$ when $\epsilon \rightarrow 0$. This contradiction proves that $1 \leq \tau < \infty$. Whence $\ln \frac{\lambda_\epsilon}{b_\epsilon^2}$ is also bounded from above by (4.12). Also by (4.12) we have

$$\ln \frac{\lambda_\epsilon}{b_\epsilon^2} \leq [4(\tau - 1)^2 + o(1)] \ln \delta - (4 + o(1)) \ln R + (64\pi^2 + o(1))\tau(\varphi(R) + 2R\varphi'(R)) + O(1)$$

which then implies $\tau = 1$ since otherwise by choosing $\epsilon > 0$ and $\delta > 0$ small enough we would obtain a contradiction with $\ln \frac{\lambda_\epsilon}{b_\epsilon^2} \geq -C_0$. With $\tau = 1$, then

$$\varphi(x) = \frac{1}{16\pi^2} \ln \frac{1}{1 + \frac{\pi}{\sqrt{6}}|x|^2}.$$

In this situation, $b_\epsilon \sim c_\epsilon$ and the estimates in (4.7) are improved as (see formula (5.21) in [30])

$$32\pi^2 \mathcal{AB}(\delta^2 - R^2 r_\epsilon^2) = o\left(\frac{1}{c_\epsilon^2}\right), \quad 32\pi^2 \mathcal{B}^2(\delta^4 - R^4 r_\epsilon^4) = o\left(\frac{1}{c_\epsilon^2}\right).$$

Consequently, (4.11) becomes

$$\begin{aligned} & \left(\frac{2\varphi(R) + R\varphi'(R) + \frac{1}{4\pi^2} \ln \delta - 2A_p - \frac{1}{8\pi^2}}{c_\epsilon^2} + \frac{\ln \frac{\lambda_\epsilon}{c_\epsilon^2} + 8 + 4 \ln \frac{R}{\delta}}{32\pi^2 c_\epsilon^2} \right) + O\left(\frac{1}{c_\epsilon^4} \ln^2 \frac{\lambda_\epsilon}{c_\epsilon^2}\right) \\ & \leq -\frac{(1 + o(1))}{c_\epsilon^2} \left(\int_{B_r(0)} |\Delta \varphi|^2 dx - \frac{1}{16\pi^2} - \frac{1}{8\pi^2} \ln \delta + A_p + O(\delta \ln \delta) \right) + o\left(\frac{1}{c_\epsilon^2}\right). \end{aligned} \quad (4.13)$$

Multiplying both sides of (4.13) by $32\pi^2 c_\epsilon^2$ we get

$$\begin{aligned} & (1 + o(1)) \ln \frac{\lambda_\epsilon}{c_\epsilon^2} \\ & \leq -32\pi^2 \left(2\varphi(R) + R\varphi'(R) + \frac{1}{4\pi^2} \ln \delta - 2A_p - \frac{1}{8\pi^2} \right) - 8 - 4 \ln \frac{R}{\delta} \\ & \quad - (32\pi^2 + o(1)) \left(\int_{B_r(0)} |\Delta \varphi|^2 dx - \frac{1}{16\pi^2} - \frac{1}{8\pi^2} \ln \delta + A_p + O(\delta \ln \delta) \right) + o(1) \\ & = -32\pi^2 (2\varphi(R) + R\varphi'(R)) - (32\pi^2 + o(1)) \int_{B_r(0)} |\Delta \varphi|^2 dx + 32\pi^2 A_p \\ & \quad - 4 \ln R - 2 + o(1)(1 - \ln \delta) + O(\delta \ln \delta) \end{aligned} \quad (4.14)$$

It was computed in [30] (see formula (5.22)) that

$$\int_{B_R(0)} |\Delta \varphi|^2 dx = \frac{1}{16\pi^2} \ln \left(1 + \frac{\pi}{\sqrt{6}} R^2 \right) + \frac{1}{96\pi^2} + O(R^{-2}).$$

Thus

$$\int_{B_R(0)} |\Delta \varphi|^2 dx = \frac{1}{8\pi^2} \ln R + \frac{1}{16\pi^2} \ln \frac{\pi}{\sqrt{6}} + \frac{1}{96\pi^2} + O(R^{-2}).$$

It is easy to see that

$$\varphi(R) = -\frac{1}{8\pi^2} \ln R - \frac{1}{16\pi^2} \ln \frac{\pi}{\sqrt{6}} + O(R^{-2}),$$

and

$$R\varphi'(R) = -\frac{1}{8\pi^2} + O(R^{-2}).$$

Plugging these estimates into (4.13), we get

$$\lim_{\epsilon \rightarrow 0} \ln \frac{\lambda_\epsilon}{c_\epsilon^2} \leq \frac{5}{3} + 32\pi^2 A_p + \ln \frac{\pi^2}{6}.$$

Thus we have proved

$$\sup_{u \in H_0^2(\Omega), \|u\|_{2,\alpha}=1} \int_{\Omega} e^{32\pi^2 u^2} dx \leq |\Omega| + \frac{\pi^2}{6} e^{\frac{5}{3} + 32\pi^2 A_p}. \quad (4.15)$$

5 Nonexistence of boundary bubbles

The main result of this section is that the boundary bubbles do not occur. Suppose without loss of generality that $c_\epsilon = u_\epsilon(x_\epsilon) = \max_{x \in \Omega} |u_\epsilon| \rightarrow \infty$ and $x_\epsilon \rightarrow p \in \partial\Omega$. Note that we have $u_\epsilon \rightharpoonup 0$ weakly in $H_0^2(\Omega)$, $u_\epsilon \rightarrow 0$ strongly in $H_0^1(\Omega)$, strongly in $L^s(\Omega)$ for any $s > 1$ and a.e., in Ω .

Lemma 5.1. *It holds $|\Delta u_\epsilon|^2 dx \rightharpoonup \delta_p$ in the sense of measure.*

Proof. Note that $\int_{\Omega} |\Delta u_\epsilon|^2 dx = 1 + \alpha \int_{\Omega} |u_\epsilon|^2 dx \rightarrow 1$. If the conclusion of this lemma is not true, then there is $r > 0$ small enough such that

$$\lim_{\epsilon \rightarrow 0} \int_{B_r(p) \cap \Omega} |\Delta u_\epsilon|^2 dx = \eta < 1.$$

Choosing ϕ be a cut-off function on $C^4(\overline{\Omega})$ such that $0 \leq \phi \leq 1$, $\phi = 1$ on $\Omega \cap B_{r/2}(p)$, $\phi = 0$ on $\Omega \setminus B_r(p)$, and $|\nabla \phi| \leq 4/r$. Since $u_\epsilon \rightharpoonup 0$ weakly in $H_0^2(\Omega)$ and $u_\epsilon \rightarrow 0$ strongly in $H_0^1(\Omega)$, hence

$$\limsup_{\epsilon \rightarrow 0} \int_{B_r(p) \cap \Omega} |\Delta(\phi u_\epsilon)|^2 dx \leq \eta.$$

This together Adams inequality and (2.3) shows that $\phi u_\epsilon \in H_0^2(\Omega)$ is weak solution of $\Delta^2(\phi u_\epsilon) = f_\epsilon$ with f_ϵ is bounded in $L^s(\Omega)$ for some $s > 1$. Applying the standard regularity theory implies that ϕu_ϵ is bounded in $C^3(\overline{\Omega})$. In particular, c_ϵ is bounded which contradicts with our assumption (3.3). \square

Lemma 5.1 proves that if there is a blow-up point on the boundary $\partial\Omega$, then this is the unique blow-up point in $\overline{\Omega}$. We next prove a convergence for $b_\epsilon u_\epsilon$.

Lemma 5.2. *It holds $b_\epsilon u_\epsilon \rightharpoonup 0$ in $H_0^{2,r}(\Omega)$ for any $1 < r < 2$.*

Proof. By the same proof of Lemma 3.3, $b_\epsilon u_\epsilon$ is bounded in $H_0^{2,r}(\Omega)$ for any $1 < r < 2$. Hence there is $F \in H_0^{2,r}(\Omega)$ such that $b_\epsilon u_\epsilon \rightharpoonup F$ in $H_0^{2,r}(\Omega)$ and $b_\epsilon u_\epsilon \rightarrow H$ in $H_0^1(\Omega)$. Using the same method in the proof of Lemma 3.4, we get that F solves $\Delta^2 F = \alpha F$ in Ω . Since $F \in H_0^{2,r}(\Omega)$ for any $1 < r < 2$, by the standard regularity theory, we have $F \in C^3(\overline{\Omega})$. However, $\alpha < \lambda_1(\Omega)$, we must have $F \equiv 0$. \square

Applying Pohozaev type identity (Lemma 3.5) to equation (2.3) on the domain $\Omega \cap B_r(p)$, we obtain by the same way in the estimates for σ^2 that

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega} e^{\alpha_\epsilon u_\epsilon^2} dx = |\Omega|$$

which contradicts with (2.5). Therefore, the blow-up point p can not lie on $\partial\Omega$.

6 Proof of Theorem 1.1

Let $c_\epsilon, x_\epsilon, p$ and A_p as before. We have shown in section §3 that if blow-up occurs, i.e., $c_\epsilon \rightarrow \infty$ then the blow-up point p lies in the interior of Ω , and the supremum

$$\sup_{u \in H_0^2(\Omega), \|u\|_{2,\alpha}=1} \int_{\Omega} e^{32\pi^2 u^2} dx \leq |\Omega| + \frac{\pi^2}{6} e^{\frac{5}{3} + 32\pi^2 A_p}. \quad (6.1)$$

We are in position to prove Theorem

Proof of Theorem. If there exists a function $u_0 \in H_0^2(\Omega)$ such that $\|u_0\|_{2,\alpha} = 1$ and

$$\int_{\Omega} e^{32\pi^2 u_0^2} dx = \sup_{u \in H_0^2(\Omega), \|u\|_{2,\alpha}=1} \int_{\Omega} e^{32\pi^2 u^2} dx,$$

then our proof is finished. Otherwise, the blow-up case occurs, hence Theorem follows from (6.1). \square

We finish this section by give a proof of Proposition 1.2 which shows that our inequality (1.8) implies the one of Lu and Yang (1.7).

Proof of Proposition 1.2. Let $a_1, a_2, \dots, a_k, k \geq 1$ be the number such that $0 \leq a_1 < \lambda_1(\Omega)$, $0 \leq a_2 \leq \lambda_1(\Omega)a_1, \dots, a_k \leq \lambda_1(\Omega)a_{k-1}$. It is easy to see that

$$q(t) \leq 1 + a_1 t + a_1 \lambda_1(\Omega) t^2 + \dots + a_1 \lambda_1(\Omega)^{k-1} t^k.$$

Denote $a = a_1/\lambda_1(\Omega) < 1$ and $p(t) = 1 + at + \dots at^k$ then

$$q(t) \leq p(\lambda_1(\Omega)t).$$

We claim that there exist $b \in (a, 1)$ such that

$$p(t) \leq \frac{1}{1 - bt}, \quad \forall t \in [0, 1]. \quad (6.2)$$

Indeed, this claim is equivalent to

$$\frac{1 - bt}{b}(1 + t + \cdots + t^{k-1}) \leq \frac{1}{a}, \quad \forall t \in [0, 1].$$

Note that

$$\frac{1 - bt}{b}(1 + t + \cdots + t^{k-1}) = \frac{1 - b}{b}(1 + t + \cdots + t^{k-1}) + 1 - t^k \leq k \frac{1 - b}{b} + 1.$$

Since $a < 1$, hence we can choose $b \in (a, 1)$ such that (6.2) holds.

Denote $\alpha = b\lambda_1(\Omega)$ with b is given in (6.2). For any $u \in H_0^2(\Omega)$ such that $\|\Delta u\|_2 \leq 1$, then $\lambda_1(\Omega)\|u\|_2^2 \leq 1$. By our claim (6.2), we have

$$q(\|u\|_2^2) \leq p(\lambda_1(\Omega)\|u\|_2^2) \leq \frac{1}{1 - \alpha\|u\|_2^2}.$$

Let

$$v = \frac{u}{(1 - \alpha\|u\|_2^2)^{1/2}},$$

then $\|v\|_{2,\alpha} \leq 1$ and $v^2 \geq q(\|u\|_2^2)u^2$. This together (1.8) implies (1.7). \square

7 Proof of Theorem 1.3

In this section, we construct functions $\phi_\epsilon \in H_0^2(\Omega)$ such that $\|\phi_\epsilon\|_{2,\alpha} = 1$ and

$$\int_{\Omega} e^{32\pi^2 \phi_\epsilon^2} dx > |\Omega| + \frac{\pi^2}{6} e^{\frac{5}{3} + 32\pi^2 A_p}.$$

This fact together (4.15) shows that the blow-up case can not occur, and hence proves our Theorem.

Denote $r = |x - p|$. Recall that

$$G_\alpha(x, p) = -\frac{1}{8\pi^2} \ln r + A_p + \psi(x), \quad \psi \in C^3(\overline{\Omega}), \psi(p) = 0.$$

Following the construction in [30] (section §7), let us define

$$\phi_\epsilon = \begin{cases} c + \frac{a - \frac{1}{16\pi^2} \ln\left(1 + \frac{\pi}{\sqrt{6}} \frac{r^2}{\epsilon^2}\right)}{c} + \frac{A_p + \psi}{c} + \frac{b}{c} r^2, & \text{if } r \leq R\epsilon, \\ \frac{1}{c} G_\alpha & \text{if } r > R\epsilon, \end{cases} \quad (7.1)$$

where a, b, c are constants determined later such that $\phi_\epsilon \in H_0^2(\Omega)$ and $\|\phi_\epsilon\|_{2,\alpha} = 1$.

We choose $R = -\ln \epsilon$. To ensure that $\phi_\epsilon \in H_0^2(\Omega)$, we choose a, b, c such that

$$\lim_{r \uparrow R\epsilon} \phi_\epsilon = \lim_{r \downarrow R\epsilon} \phi_\epsilon, \quad \lim_{r \uparrow R\epsilon} \nabla \phi_\epsilon = \lim_{r \downarrow R\epsilon} \nabla \phi_\epsilon.$$

The simple computation shows that

$$\begin{cases} a = -c^2 + \frac{1}{16\pi^2} \ln \left(1 + \frac{\pi}{\sqrt{6}} R^2 \right) - \frac{\ln(R\epsilon)}{8\pi^2} - bR^2\epsilon^2, \\ b = -\frac{1}{16\pi^2 R^2 \epsilon^2 \left(1 + \frac{\pi}{\sqrt{6}} R^2 \right)}. \end{cases} \quad (7.2)$$

It was computed in [30] that

$$\|\Delta \phi_\epsilon\|_2^2 = \frac{1}{16\pi^2 c^2} \left(\ln \frac{\pi}{\sqrt{6}\epsilon^2} + 16\pi^2 A_p - \frac{5}{6} \right) + \frac{\alpha}{c^2} \|G_\alpha\|_2^2 + O\left(\frac{1}{c^2 \ln^2 \epsilon}\right).$$

We have

$$\int_{\Omega \setminus B_{Rr\epsilon}(p)} \phi_\epsilon^2 dx = \frac{1}{c^2} \int_{\Omega \setminus B_{Rr\epsilon}(p)} G_\alpha^2 dx = \frac{1}{c^2} \|G_\alpha\|_2^2 - \frac{1}{c^2} \int_{B_{Rr\epsilon}(p)} G_\alpha^2 dx = \frac{1}{c^2} \|G_\alpha\|_2^2 + \frac{O(\epsilon^4 \ln^6(\epsilon))}{c^2}.$$

On $B_{R\epsilon}(p)$ we have

$$\begin{aligned} \phi_\epsilon(x) &= \frac{1}{16\pi^2 c} \left(\ln \left(1 + \frac{\pi}{\sqrt{6}} R^2 \right) - \ln \left(1 + \frac{\pi}{\sqrt{6}} \frac{r^2}{\epsilon^2} \right) \right) - \frac{\ln(R\epsilon)}{8\pi^2 c} \\ &\quad + \frac{A_p + \psi}{c} - \frac{b}{c} R^2 \epsilon^2 \left(1 - \frac{r^2}{R^2 \epsilon^2} \right), \end{aligned}$$

hence

$$\int_{B_{R\epsilon}(p)} \phi_\epsilon^2 dx = \frac{1}{c^2} O(\epsilon^4 \ln^6 \epsilon).$$

Combining all these estimates together, we get

$$\|\phi_\epsilon\|_{2,\alpha}^2 = \frac{1}{16\pi^2 c^2} \left(\ln \frac{\pi}{\sqrt{6}\epsilon^2} + 16\pi^2 A_p - \frac{5}{6} \right) + O\left(\frac{1}{c^2 \ln^2 \epsilon}\right).$$

Thus we can choose c such that $\|\phi_\epsilon\|_{2,\alpha} = 1$ for ϵ small enough. Moreover, we have

$$c^2 = \frac{1}{16\pi^2} \left(\ln \frac{\pi}{\sqrt{6}\epsilon^2} + 16\pi^2 A_p - \frac{5}{6} \right) + O\left(\frac{1}{\ln^2 \epsilon}\right). \quad (7.3)$$

We next compute $\int_\Omega e^{32\pi^2 \phi_\epsilon^2} dx$. On $\Omega \setminus B_{R\epsilon}(p)$ we have

$$\begin{aligned} \int_{\Omega \setminus B_{R\epsilon}(p)} e^{32\pi^2 \phi_\epsilon^2} dx &\geq \int_{\Omega \setminus B_{R\epsilon}(p)} \left(1 + \frac{32\pi^2}{c^2} G_\alpha^2 \right) dx \\ &= |\Omega| + \frac{32\pi^2}{c^2} \|G_\alpha\|_2^2 + O\left(\frac{1}{\ln^2 \epsilon}\right). \end{aligned} \quad (7.4)$$

On $B_{R\epsilon}(p)$, using (7.2) and (7.3) we have

$$\begin{aligned}
\phi_\epsilon^2 &\geq c^2 + 2 \left(a - \frac{1}{16\pi^2} \ln \left(1 + \frac{\pi}{\sqrt{6}} \frac{r^2}{\epsilon^2} \right) + A_p + \psi + br^2 \right) \\
&= -c^2 + 2(a + c^2) - \frac{1}{8\pi^2} \ln \left(1 + \frac{\pi}{\sqrt{6}} \frac{r^2}{\epsilon^2} \right) + 2A_p + 2\psi + 2br^2 \\
&= -\frac{1}{16\pi^2} \ln \frac{\pi}{\sqrt{6}\epsilon^2} + \frac{5}{96\pi^2} + \frac{1}{8\pi^2} \ln \left(1 + \frac{\pi}{\sqrt{6}} R^2 \right) - \frac{\ln(R\epsilon)}{4\pi^2} \\
&\quad - \frac{1}{8\pi^2} \ln \left(1 + \frac{\pi}{\sqrt{6}} \frac{r^2}{\epsilon^2} \right) + A_p + O \left(\frac{1}{\ln^2 \epsilon} \right),
\end{aligned}$$

here we use the fact $\psi(p) = 0$, hence $\psi = O \left(\frac{1}{\ln^2 \epsilon} \right)$ on $B_{R\epsilon}(p)$ since $R = -\ln \epsilon$ and also $br^2 = O \left(\frac{1}{\ln^2 \epsilon} \right)$ on $B_{R\epsilon}(p)$. Hence, on $B_{R\epsilon}(p)$, we have

$$\begin{aligned}
e^{32\pi^2 \phi_\epsilon^2} &\geq \left(\frac{\pi^2}{6\epsilon^4} \right)^{-1} e^{\frac{5}{3} + 32\pi^2 A_p} \left(1 + \frac{\pi}{\sqrt{6}} R^2 \right)^4 (R\epsilon)^{-8} \left(1 + \frac{\pi}{\sqrt{6}} \frac{r^2}{\epsilon^2} \right)^{-4} \left(1 + O \left(\frac{1}{\ln^2 \epsilon} \right) \right) \\
&= \frac{\pi^2}{6} e^{\frac{5}{3} + 32\pi^2 A_p} \epsilon^{-4} \left(1 + \frac{\pi}{\sqrt{6}} \frac{r^2}{\epsilon^2} \right)^{-4} \left(1 + O \left(\frac{1}{\ln^2 \epsilon} \right) \right),
\end{aligned}$$

since $R = -\ln \epsilon$. Integrating on $B_{R\epsilon}(p)$ and using a suitable change of variable, we get

$$\int_{B_{R\epsilon}(p)} e^{32\pi^2 \phi_\epsilon^2} dx \geq \left(1 + O \left(\frac{1}{\ln^2 \epsilon} \right) \right) e^{\frac{5}{3} + 32\pi^2 A_p} \int_{B_{\bar{R}}(0)} (1 + |x|^2)^{-4} dx.$$

with $\bar{R} = \pi^{1/2} R / 6^{1/4}$. Using polar coordinate we get

$$\begin{aligned}
\int_{B_{\bar{R}}(0)} (1 + |x|^2)^{-4} dx &= 2\pi^2 \int_0^{\bar{R}} \frac{r^3}{(1 + r^2)^4} dr \\
&= \pi^2 \int_0^{\bar{R}^2} \frac{r}{(1 + r)^4} dr \\
&= \pi^2 \left(\frac{1}{6} - \frac{1}{2(1 + \bar{R}^2)^2} + \frac{1}{3(1 + \bar{R}^2)^3} \right) \\
&= \frac{\pi^2}{6} \left(1 + O \left(\frac{1}{\ln^4 \epsilon} \right) \right).
\end{aligned}$$

Finally, we have

$$\int_{B_{R\epsilon}(p)} e^{32\pi^2 \phi_\epsilon^2} dx \geq \frac{\pi^2}{6} e^{\frac{5}{3} + 32\pi^2 A_p} + O \left(\frac{1}{\ln^2 \epsilon} \right). \quad (7.5)$$

Combining (7.4) together (7.5) we obtain

$$\int_{\Omega} e^{32\pi^2 \phi_\epsilon^2} dx \geq |\Omega| + \frac{\pi^2}{6} e^{\frac{5}{3} + 32\pi^2 A_p} + \frac{32\pi^2}{c^2} \|G_\alpha\|_2^2 + O \left(\frac{1}{\ln^2 \epsilon} \right).$$

This together (7.3) imply that for ϵ is sufficiently small

$$\int_{\Omega} e^{32\pi^2\phi_{\epsilon}^2} dx > |\Omega| + \frac{\pi^2}{6} e^{\frac{5}{3}+32\pi^2 A_p},$$

as our desire.

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